# The Mathematics of Sinking Through Sharply Stratified Liquids

An Experimental Approach with Spheres

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## Abstract

**Scientists have observed sharp stratification as a key process in the ocean's natural cycles, and more extensively, the Earth's. Sharp stratification is a known phenomenon in liquid dynamics that is still not well understood. This paper focuses on filling that void by observing the experimental behavior of particles that sink through multiple layers and mathematically analyzing it. Specifically, this paper will be utilizing a two-spheres, two-layer regime. Of crucial importance to this paper is the newly discovered phenomenon (by UNC graduates) of flow stagnant points induced by the stratification. This thesis is heavily built upon experiments, and is motivated by the uniting of this data to the mathematics that govern it. Ultimately, this thesis should act as a stepping stone for future work.**

## Acknowledgments

None of this project would have been possible if not for my amazing adviser, Roberto Camassa. I must have approached him only a few days before the deadline for submitting a thesis proposal. I had no adviser and no idea of what I was interested in studying. By some spark of insanity, Professor Camassa agreed to help me out, a true testament to his character and interest in the academic ambitions of his students. As busy of a man as Professor Camassa is, it amazes me how he somehow found the time to meet with me, direct my studies and keep a good tally of my progress (an impressive feat for a man who juggles a small army of graduates). I owe this project to his intelligent and caring guidance. Thank you for everything professor!

In the summer before I started this thesis, I vividly remember a conversation I had with a graduate student. I asked her about the thesis. What is the honors thesis? Is it worth my time? And am I even capable enough to finish it? To which she responded, to the best of my memory, "Dylan, you should definitely do it! And if it doesn't work out, which can't even happen, you will still learn a lot from it. And don't worry, I'll be there if you need any help." That graduate student was Claudia Falcon and that advice gave me the confidence I needed to start this long endeavor and difficult undertaking. Every step of the way, she was always there to offer a helping hand. This testimony doesn't do her any justice, especially in consideration of the many long hours she would spend with me weekly. Thank you Claudia for everything! If you are ever in need, I will be there as a colleague and a close friend.

I also must mention Professor Richard McLauglin and Professor Daniel Harris. Having no real obligation to help me, they found the time in there busy schedules. You guys are awesome!

Lastly, I should thank Claudia's little helpers Sabina Iftikhar and Gabriella Stein. Although I can't overemphasize how many times they helped me in my experiments, their largest contribution, by far, was their great personalities and friendship. They made the lab a fun place to work, and helped me get through the grind of running these long experiments.

To all of you, I feel unjustified in representing all that you have done for me in a few sentences, hidden among one hundred other pages of many more sentences. Just know that these words are just a representation of how much I appreciate you all, and no finite collection of sentences I could fit on this page, could ever come close to describing that.

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## Contents

## 2 Introduction

Density stratification is a key process on Earth. It manifests in many natural forms, from governing pollutants in the atmosphere, to oil leaks in the ocean. However, one of the most important and delicate mechanisms that it governs is the carbon cycle of the ocean. As global temperatures rise, the process by which the ocean recycles carbon gas into the atmosphere is changing in currently unpredictable and detrimental ways. In this paper, we will better understand this phenomenon by studying the mathematics that govern the descending of glass spheres in a stratified regime, a substitute for marine aggregates/ marine snow in the ocean (a crucial part of the carbon cycle). Of particular interest is the low-Reynolds two-spheres, two-layers regime. This thesis will follow the path of my research. Beginning with a literature review as a basis for my work, the paper will progress from rudimentary mathematical observation, to experiments. The progression of experiments will be in this order: two spheres in water, two spheres in homogenous corn syrup, and lastly two spheres in stratified corn syrup. The paper will be brought full circle with mathematical conclusions from the analyzed observations.

## 3 Literature Review

**Up to this point, there have been great strides in empirically predicting a rigid sphere falling in a homogeneous regime. However, due to the currently unsolved 3-dimensional Navier-Stokes equations, the coefficient of drag has only been empirically derived or approximated with solvable simplifications of Stokes Flow (which is a simplification itself). These text explore that derivation:**

**(Clift, R.,Grace, J.R., and Weber, M.E., 1978)**- "Bubbles,Drops, and Particles" is a comprehensive study on predicting rigid objects in low Reynolds regimes. The research, compiled by three chemical engineers, discusses the numerous drag coefficients that had been derived over the years, and offers a new empirically derived equation for the coefficient.

**(Morrison,F.A. 2013)** Faith Morrison, a chemical engineer,used a larger source of empirical data to create a more recent, and accurate best fit model of her own. She created hers with the intent to develop a robust formula that could approximate the drag coefficient "over the entire Reynolds-number range of the available experimental data", and is developed, then analyzed in this paper.

**For predicting the velocities of two spheres in homogeneous low Reynolds solutions there are a number of early papers that provide analytic approximations which they compare to their experimental data:**

**(Kynch, G. J. 1958)** Kynch derives and approximate model for the velocity of two spheres in low Reynolds as a function of separation and radii. He corroborates his model with his own experiments, and finds the match "favorable". Most importantly, he states "Equal spheres falling under gravity fall together with a constant separation, but they only fall vertically when the line of centres is either horizontal or vertical; otherwise they tend to slide downwards along the line of the centres."

**(Davies, G. A. and, Rushton, E., 1973)** They try to build upon earlier work, like Kynch's, by simplifying the situation further and limiting their configurations for the two spheres, specifically to the co-axial (snowman/stacked) case. They develop a general solution for creeping flow, and believe it agrees with solutions previously published.

**For sharply stratified regimes, a lot of the research comes from UNC's Fluids Lab department:**

**(Abaid, N., Adalsteinsson, D., Agyapong, A. and McLaughlin, R. M.,2004)**
One of the first analysis into the "floating" effect of a falling sphere due to the entrainment around a sinking sphere of lower density top layer fluids, in a more dense, bottom layer fluid. This paper acts as a starting point for my research into the properties and equations that govern a sphere descending in a sharply stratified low Reynolds regime.

**(Roberto, C., Falcon, C., Lin, J., McLaughlin, R., Mykins, N., 2009)**
A paper developed by UNC Graduates and Professors to create a first-principle prediction of a sphere falling in low Reynolds through a stratified

regime. With the help of a Green's function formulation, they develop a numerical model of the forces caused by the entrainment of a sphere, and then compare there computational method to experiments ran in the lab. Of particular interest is their investigation into the velocity fields created by the entrainment.

**This Paper**
This paper is a progression and, ultimately, a union of the aforementioned papers. Attempting to wed the models of two sphere configurations with the effect of entrainment. This paper also reports the first recorded observance of the approach of two spheres in the bottom layer of a sharply stratified due to the force of entrainment.

# 4 Terminal Velocity of a Single Sphere in Low Reynolds Free Space

A single sphere's fall is governed by three forces, Gravity, Buoyancy and Drag:

**(Gravity and Buoyancy)** $\frac{4}{3}\pi r^3(\rho_{particle} - \rho_{fluid})g$

**(Drag)** $\frac{1}{2}\pi r^2 C_d \rho_{fluid} V^2$

Likewise Newton's second law:

**(Total Force** $= Mass \cdot Acceleration$**)** $-\frac{4}{3}\pi r^3 \rho_{particle} \frac{dV}{dt}$

These forces are collectively represented in the equation:

$$\frac{4}{3}\pi r^3 \rho_{particle} \frac{dV}{dt} = \frac{4}{3}\pi r^3(\rho_{particle} - \rho_{fluid})g + sign(V)\frac{1}{2}\pi r^2 C_d \rho_{fluid} V^2$$

(Equation 1)

Where $V$ =Velocity, $t$ =Time, $\mu$ =Dynamic Viscosity, $r$ =Radius, $\rho$ =Density, and $g$ =Gravitational Constant.

We can solve for the terminal velocity at low Reynolds ($Reynolds \ll 1$) by first replacing our coefficient of friction ($C_d$) with the Reynolds dependent function for stoke flow $24/Re$. Then since terminal Velocity is a constant, $\frac{dV}{dt} = 0$, which makes the left hand side of (equation 1) zero. Lastly, by moving and collecting terms we get:

$$V_{terminal} = \frac{2r^2(\rho_{fluid} - \rho_{particle})g}{9\mu}$$

(Equation 2)

# 5 Elementary models to predict two spheres falling:

Suppose we drop two identical spheres lined horizontally and tangential (symmetry and low Reynolds reversibility should guarantee there velocities are identical), where $V_{t1}$ is the terminal velocity of one of our spheres and $V_{t2}$ is the terminal velocity of both our spheres with the respective assumption:

**(1)What if we just treat this as a sphere with twice the radius?**

$$V_{t1} = \frac{2r^2(\rho_{fluid} - \rho_{particle})g}{9\mu}$$

$$V_{t2} = \frac{2(2r)^2(\rho_{fluid} - \rho_{particle})g}{9\mu}$$

$$= \frac{8r^2(\rho_{fluid} - \rho_{particle})g}{9\mu}//$$

$$= 4(\frac{2r^2(\rho_{fluid} - \rho_{particle})g}{9\mu})$$

$$\Rightarrow 4V_{t1} = V_{t2}$$

**(2)What if we just treat this as a sphere with twice the volume?**

"Volume of a single sphere"$=v_1=\frac{4}{3}\pi r_1^3 \Rightarrow r_1 = \frac{3v_1}{4\pi}^{1/3}$

"Volume of hypothetical sphere"$=v_2=2v_1=\frac{4}{3}\pi r_2^3$

$\Rightarrow r_2 = \frac{3(2v_1)}{4\pi}^{1/3} = 2^{1/3}(\frac{3v_1}{4\pi}^{1/3})$

$\Rightarrow 2^{1/3}r_1 = r_2$ (Substitution)

$\Rightarrow 2^{2/3}V_{t1} = V_{t2}$ (Same steps as in (1))

**(3) What if we just treat this as a sphere with twice the radius and compensate for the density by calculating the density of a sphere with twice the radius that inscribes the two particles and fill the rest with the surrounding fluid?**

Lets first concentrate on how changing the density calculation effects our terminal velocity.

$$V_{t1} = \frac{2r^2(\rho_{fluid} - \rho_{particle1})g}{9\mu}$$

For the term $(\rho_{fluid} - \rho_{particle1})$ of our second particle:

$$\rho_{particle_2} = \rho_{particle1}\frac{V_{twoParticles}}{V_{total}} + \rho_{fluid}\frac{V_{fluid}}{V_{total}}$$

$= \frac{\rho_{particle1}}{4} + \frac{3\rho_{fluid}}{4}$ (Simple volume calculation)

$$V_{t2} = \frac{2(2r)^2(\rho_{fluid} - (\frac{\rho_{particle1}}{4} + \frac{3\rho_{fluid}}{4})g}{9\mu}$$

$$= 4(\frac{2r^2(\frac{\rho_{fluid} - \rho_{particle1}}{4})g}{9\mu})$$

$$= (4 \cdot \frac{1}{4})\frac{2r^2(\rho_{fluid} - \rho_{particle1})g}{9\mu}$$

$\Rightarrow V_{t1} = V_{t2}$(Substitution)

**(4)What if we just treat this as a sphere with twice the mass?**

Since $\rho$ =mass/volume $\Rightarrow \rho_{particle_2} = 2\rho_{particle_1}$
Thus: $V_{t2} = \dfrac{2r^2(\rho_{fluid} - 2\rho_{particle})g}{9\mu}$
$= \dfrac{2r^2(\rho_{fluid} - \rho_{particle})g}{9\mu} + \dfrac{2r^2(\rho_{particle})g}{9\mu}$
$\Rightarrow V_{t1} + \dfrac{2r^2 g}{9\mu}\rho_{particle} = V_{t2}$

These naive models give us a good understanding of some possible upper bounds, and certainly suggest that two spheres will sink faster then one.

# 6 Developing a Drag Coefficient for Finite Reynolds

The crucial, and Reynolds dependent, drag coefficient of our terminal velocity equation has been approximated by various calculations. Although these approximations are nearly identical in Low Reynolds regimes (as they should all converge to Stokes), they tend to diverge from each other at $Re > 1$. To begin, the simplest and easiest drag coefficient to calculate is just the stokes defined drag coefficient. Since this calculation is straightforward, I will quickly walk us through it:

$$F_d = 3\pi\mu U d \textbf{(Stokes Definition for Drag Force)}$$

and

$$C_d = \frac{2F_d}{\rho U^2 A}\textbf{(Definition of Drag Coefficient)}$$

$$\Rightarrow C_d = \frac{\mu 24}{\rho U d}$$

Since, $Re = \dfrac{\rho U d}{\mu}$

$$\Rightarrow C_d = \frac{24}{Re}$$

Where, d is our sphere's diameter, $A = \pi d^2/4$ is the cross sectional area of our sphere, $\mu$ is are dynamic viscosity, $\rho$ is our fluid density, and U is our fluid velocity (relative to sphere)

Stokes drag is accurate only for low-Reynolds flow (at higher Reynolds, stokes equation makes thing impossible to calculate analytically).

In the Early 1900's, Oseen built upon Stokes calculation by correcting for an originally neglected inertial effect and derived the following approximation:

$$\frac{24[1 + 3Re/16]}{Re}$$

Then in 1978 R. Clift, J. Grace, and M. Weber, M.E., did a comprehensive study into all the work that had been produced to approximate the coefficient of drag, across the range of feasible Reynolds values. They even provided a table of the many empirically derived coefficients of drag.

Then presented their own which represented a larger span of empirical data $1 < Re < 1000$:

$$C_d = \frac{24[1 + 0.15Re^0.687}{Re}$$

(Clift et al., 1978)

What I am trying to show here is that for a single, rigid particle in a homogeneous regime, the fluid dynamics are pretty well understood. The big complication comes from approximating the drag coefficient, which, until the stokes equation is solved, will be empirically derived. That being said, as more experiments are performed and more data becomes accessible, more accurate approximations will be derived. In recent years Morrison has developed her own. Since we are beginning our experiments in water, this will be especially relevant, so we will look into her equation in a later section.

TABLE 5.1

Relationships for Sphere Drag

| | Author(s) | Range | Relationship for $C_D$ | Range of deviation in $C_D$ (%) |
|---|---|---|---|---|
| 1. | Schiller and Nauman (S1) | $\mathrm{Re} < 800$ | $\frac{24}{\mathrm{Re}}(1 + 0.15\,\mathrm{Re}^{0.687})$ | $+5$ to $-4$ |
| 2. | Lapple (L3) | $\mathrm{Re} < 1000$ | $\frac{24}{\mathrm{Re}}(1 + 0.125\,\mathrm{Re}^{0.72})$ | $+5$ to $-8$ |
| 3. | Langmuir and Blodgett (L2) | $1 < \mathrm{Re} < 100$ | $\frac{24}{\mathrm{Re}}(1 + 0.197\,\mathrm{Re}^{0.63} + 2.6 \times 10^{-4}\,\mathrm{Re}^{1.38})$ | $+6$ to $+1$ |
| 4. | Allen (A5) | (a) $2 < \mathrm{Re} < 500$<br>(b) $1 < \mathrm{Re} < 1000$ | $10\,\mathrm{Re}^{-1/2}$<br>$30\,\mathrm{Re}^{-0.625}$ | $-8$ to $-52$<br>$+70$ to $-15$ |
| 5. | Gilbert *et al.* (G7) | $0.2 < \mathrm{Re} < 2000$ | $0.48 + 28\,\mathrm{Re}^{-0.85}$ | $+24$ to $-11$ |
| 6. | Kurten *et al.* (K8) | $0.1 < \mathrm{Re} < 4000$ | $0.28 + \frac{6}{\mathrm{Re}^{1/2}} + \frac{21}{\mathrm{Re}}$ | $+7$ to $-6$ |
| 7. | Abraham (A2) | $\mathrm{Re} < 6000$ | $0.2924(1 + 9.06\,\mathrm{Re}^{-1/2})^2$ | $+9$ to $-6$ |
| 8. | Ihme *et al.* (I1) | $\mathrm{Re} < 10^4$ | $0.36 + \frac{5.48}{\mathrm{Re}^{0.573}} + \frac{24}{\mathrm{Re}}$ | $+10$ to $-10$ |
| 9. | Rumpf [see (K8)] | (a) $\mathrm{Re} < 10$<br>(b) $\mathrm{Re} < 100$<br>(c) $\mathrm{Re} < 10^5$ | $2 + 24/\mathrm{Re}$<br>$1 + 24/\mathrm{Re}$<br>$0.5 + 24/\mathrm{Re}$ | $-3$ to $-5$<br>$+14$ to $-20$<br>$+30$ to $-39$ |
| 10. | Clift and Gauvin (C6) | $\mathrm{Re} < 3 \times 10^5$ | $\frac{24}{\mathrm{Re}}(1 + 0.15\,\mathrm{Re}^{0.687}) + 0.42/(1 + 4.25 \times 10^4\,\mathrm{Re}^{-1.16})$ | $+6$ to $-4$ |
| 11. | Brauer (B11) | $\mathrm{Re} < 3 \times 10^5$ | $0.40 + \frac{4}{\mathrm{Re}^{1/2}} + \frac{24}{\mathrm{Re}}$ | $+20$ to $-18$ |
| 12. | Tanaka and Iinoya (T1) | $\mathrm{Re} < 7 \times 10^4$ | $\log_{10} C_D = a_1 w^2 + a_2 w + a_3$ where $w = \log_{10}\mathrm{Re}$ and $a_1$, $a_2$, and $a_3$ are given for 7 intervals of Re | $+6$ to $-9$ |

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Figure 1: An example of the wide variety of drag coefficients developed that emphasizes how important this parameter is, and how difficult it is to derive

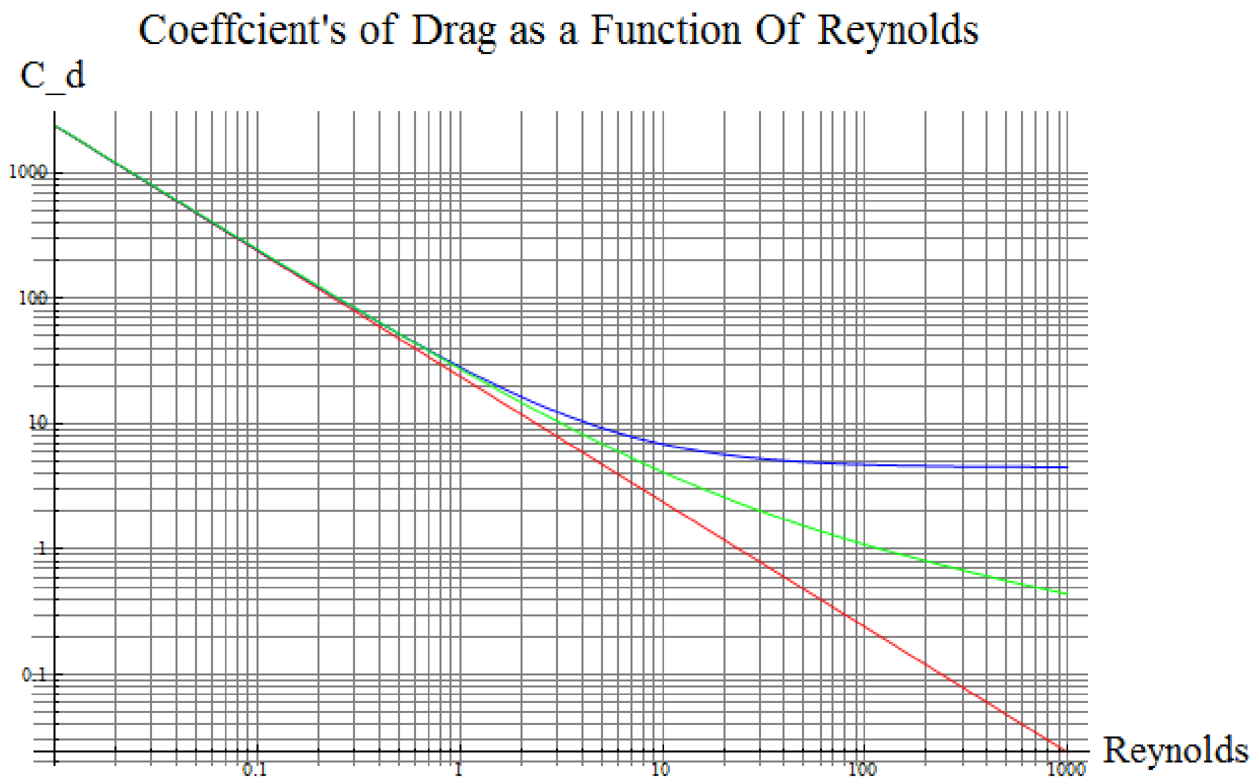



Figure 2: The Oseen in Blue and Stokes in Red, in general, act as a upper and lower bound for these empirically derived equations, (Clift, et al.) in Green

# 7 Terminal Velocity of a Single Sphere Finite Reynolds

To continue this investigation, we need to be aware that we are operating on the cusp of what many believe to be low Reynolds versus finite Reynolds. What differentiates the two is the complexity and magnitude of the coefficient of drag. In the low Reynolds case our coefficient of drag was $24/Re$, which simplified equation 1 to equation 2, but at this point, it may be more accurate to use different coefficients.

Dr. Faith Morrison from M.I.T. in her paper "Data Correlation for Drag Coefficient for Sphere" develops a equation for the drag coefficient that uses data correlations from experiments to provide a reasonably accurate data set from low Reynolds flow to high. More specifically, she cites her equations being accurate for low-Reynolds all the way up to Reynolds of $10^7$, covering a large spectrum of the available experimental data. However, as shown in the model graph below, the equation for the drag coefficient is actually piecewise. For $Re < 2$ the drag coefficient is $24/Re$ (creeping Reynolds), the stokes coefficient for low Reynolds. For the range of $1 < Re < 10^6$ (recirculating) it uses the provided, empirically derived, drag coefficient. Lastly, for $10^6 < Re < 10^7$ (turbulent) the equation models uniform flow around a sphere with "a line with slope of 0.80 on a log-log graph".

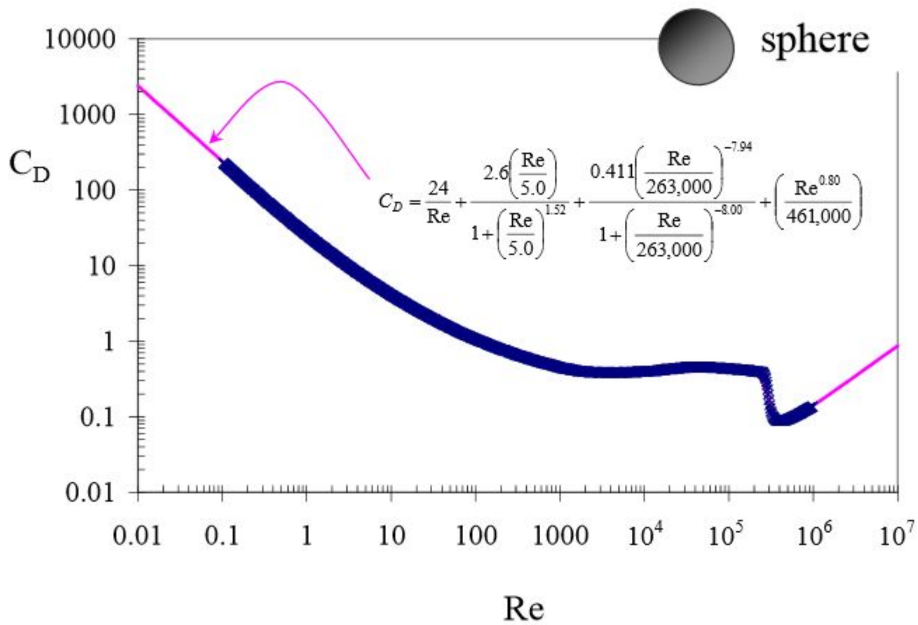
To recap, equation 1 for Reynolds:

$$\frac{4}{3}\pi r^3 \rho_{particle}\frac{dV}{dt} = \frac{4}{3}\pi r^3(\rho_{particle} - \rho_{fluid})g - sign(V)\frac{1}{2}\pi r^2 C_d \rho_{fluid} V^2$$

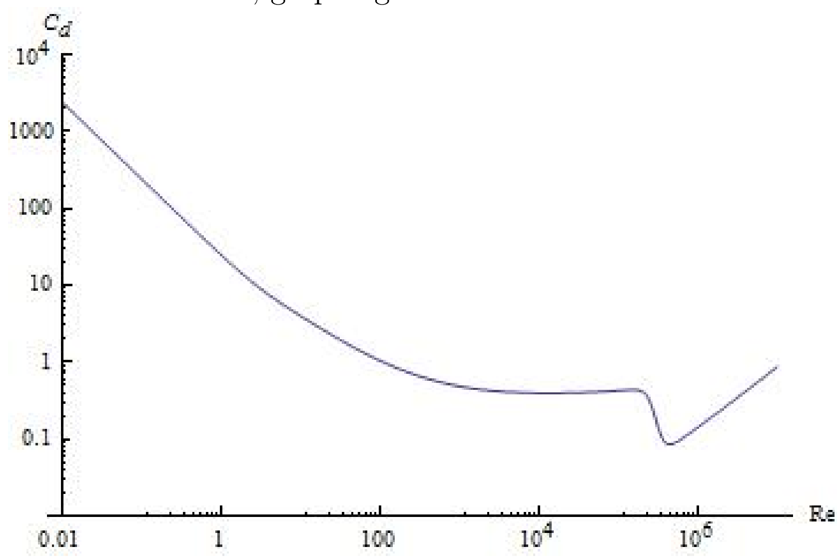
Where Morrison's coefficient is:

$$C_d = \frac{24}{Re} + \frac{2.6(\frac{Re}{5})}{1+(\frac{Re}{5})^{1.52}} + \frac{.411(\frac{Re}{263000})^{-7.94}}{1+(\frac{Re}{263000})^{-8}} + \frac{Re^{0.8}}{461000}$$

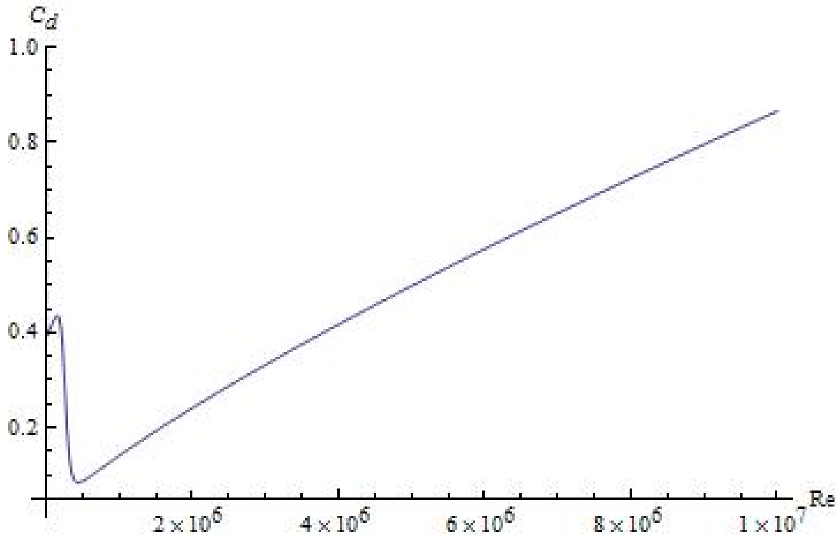
Graphing Reynolds versus this coefficient from $.01 < Re < 10^7$ Morrison got this log-log graph:



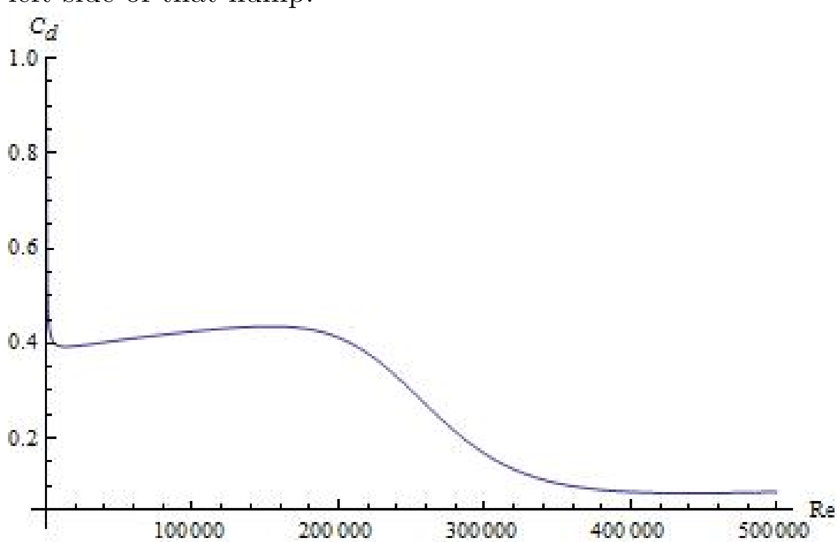

Just to make sure, graphing it ourselves:



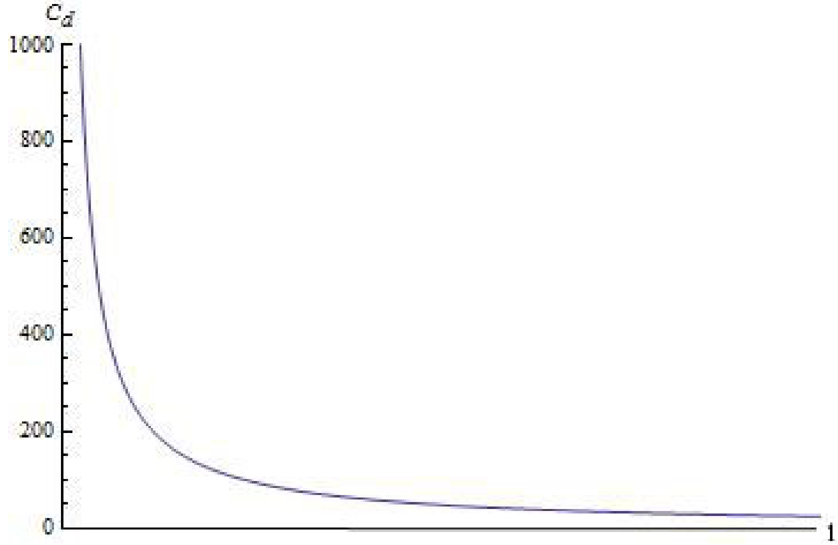

Unfortunately, this rendering really prevents us from getting a strong understanding of this behavior due to its logarithmic data points that hide its true shape. Thus lets look at it with a scaled axis ranging over the same points:




Now we can observe how for very large Reynolds values, the slope is almost linear, and less then 1 (drag always reduces velocity). Lets look closer at the left side of that hump:




That hump seems to really be a bunch of points cluttered around .4, but notice that slight climb on the left of the graph. Lets look closer at this and see if this is where our colossal range on our original log-log graph originated.




So, it appears that the coefficient asymptotically approaches infinity on the y axis, this explains the high numbers for low Reynolds. But what does it approach for certain? Although Morrison advises to use $24/Re$ for low Reynolds, we should take a moment to see how her experimental coefficient acts around 0.

# 8 Taylor Approximations:Morrison's Coefficient As Reynolds Approaches Zero

**Examining How the Equation Acts:**

We need to move all parts of this equation that are functions of Reynolds to one side of the equation so that we can observe the behavior as Reynolds approaches zero.

Since we are focusing on terminal velocity, the left hand side of:

$$\frac{4}{3}\pi r^3 \rho_{particle}\frac{dV}{dt} = \frac{4}{3}\pi r^3(\rho_{particle} - \rho_{fluid})g - sign(V)\frac{1}{2}\pi r^2 C_d \rho_{fluid} V^2$$

becomes zero, then neglecting the $sign(V)$ term (focused on magnitude), and reorganizing the other two terms, gives us:

$$\frac{8r(\rho_{particle} - \rho_{fluid})g}{3\rho_{fluid}} = V^2 C_d$$

Since velocity can be rewritten as a function of Reynolds:

$$Re = \frac{V r \rho_{fluid}}{\mu} \Rightarrow V^2 = (\frac{Re\mu}{r\rho_{fluid}})^2 \text{ (Substituting into our equation)}$$
$$\Rightarrow \frac{8r^3\rho_{fluid}(\rho_{particle} - \rho_{fluid})g}{3\mu^2} = Re^2 C_d$$

Now Concentrating on the right side. We get:

$$Re^2 C_d = 24Re + \frac{2.6(\frac{Re^3}{5})}{1 + (\frac{Re}{5})^{1.52}} + \frac{.411(\frac{Re}{263000})^{-5.94}}{1 + (\frac{Re}{263000})^{-8}} + \frac{Re^{2.8}}{461000}$$

To simplify this further lets expand these term by there Taylor series at 0, up to order $O[Re]^3$.

For the first term $24Re$ it is it's own Taylor series, so we will leave as is:

$$24Re$$

**(First Term)**

For the second term $\frac{2.6(\frac{Re^3}{5})}{1 + (\frac{Re}{5})^{1.52}}$, we will make a adjustment by assuming $(Re/5)^{1.52} \approx (Re/5)^{1.5}$. Now by substituting the powers we get:

$$\frac{325(\frac{Re^{3/2}}{5^{3/2}}^2)}{1 + (\frac{Re^{3/2}}{5^{3/2}})}$$

Taylor expanding this series where $x = \frac{Re^{3/2}}{5^{3/2}}$:

$$325(x^2 - x^3 + x^4 - x^5 + ...)$$

Truncating at $O[Re]^3 \Rightarrow O[x^2]$ gives us:

$$325(x^2) \Rightarrow 2.6Re^3$$

**(Second Term)**

For the third term $\frac{.411(\frac{Re}{263000})^{-5.94}}{1+(\frac{Re}{263000})^{-8}}$, we will make a adjustment by assuming $(Re/263000)^{-5.94} \approx (Re/263000)^{-6}$. Now substituting the powers we get:

$$\frac{.411 * 263000^2(\frac{Re}{263000})^{-6}}{1+(\frac{Re}{263000})^{-8}}$$

Taylor expanding this series where $x = \frac{Re}{263000}$:

$$.411(263000^2)(x^2 - x^10 + x^18 - x^26 + ...)$$

Truncating at $O[Re]^3 \Rightarrow O[x^3]$ gives us:

$$.411(263000)(x^2) \Rightarrow .411Re^2$$

**(Third Term)**

Lastly, for the forth term: $\frac{Re^{2.8}}{461000}$, we will make a adjustment by assuming $(Re)^{2.8} \approx (Re)^3$. Now we end up with $\frac{Re^3}{461000}$, which would combine with our second term. But since the coefficient $1/461000 = 2.169 * 10^{-6} \ll 2.6$ (the coefficient of the second term),we will neglect this term.

Thus our final Taylor Approximation for Reynolds around 0 is:

$$\Rightarrow \frac{8r^3\rho_{fluid}(\rho_{particle} - \rho_{fluid})g}{3\mu^2} = 24Re + .411Re^2 + 2.6Re^3$$

(equation 4)
As expected, Reynolds approaching zero, makes the right side of our equation zero. Solving it implies that $\rho_{fluid} = \rho_{particle}$, which we know implies

that the particle would remain stationary in the liquid. This means that velocity at this point is also zero. As expected once you consider that $0 = Re = \dfrac{Vr\rho_{fluid}}{\mu}$, since the density of the fluid and the radius of the particle are non zero for any given scenario, the velocity must be zero. This then agrees with our conclusion.

**Examining how $C_d$ acts**

To observe $C_d$'s behavior we nullify multiplying through by the $Re^2$ contributed by the $V^2$ term by dividing the RHS of equation 4 by $Re^2$, giving us:

$$24/Re + .411 + 2.6Re = C_d$$

which approaches infinity as Reynolds approaches zero. This is in agreement with our graph.

# 9 Boundary Condition

Before we begin our experiments, there remains one essential caveat for predicting the terminal velocity; a boundary condition to account for not being in free space (these experiments will be conducted in cylindrical tanks). This boundary condition has a well known derivation [available in (Camassa et. al.) paper if curious), that uses the Faxen's law. Faxen's law, developed much later in this paper, relates the forces of sphere from the flow in the tank. This addition can compensate for the addition of walls using the "method of reflections". This compensation for flow using Faxen's Law is called the "Faxen correction". For Newtonian fluids, a sphere falling through a cylindrical tank will have the following Faxen Correction (where $R$ is radius and $a$ is the radius of the sphere:

$$K_N(\frac{a}{R}) = [1 - 2.10444(\frac{a}{R}) + 2.08877\frac{a}{R}^3 - 0.94813(\frac{a}{R})^5 - 1.372(\frac{a}{R})^6 + 3.87(\frac{a}{R})^8 - 4.19(\frac{a}{R})^{10} + ...]^{-1}$$

For our purposes, we will only need this to order 3.

# 10 Experiment Setup and Analysis:Visual

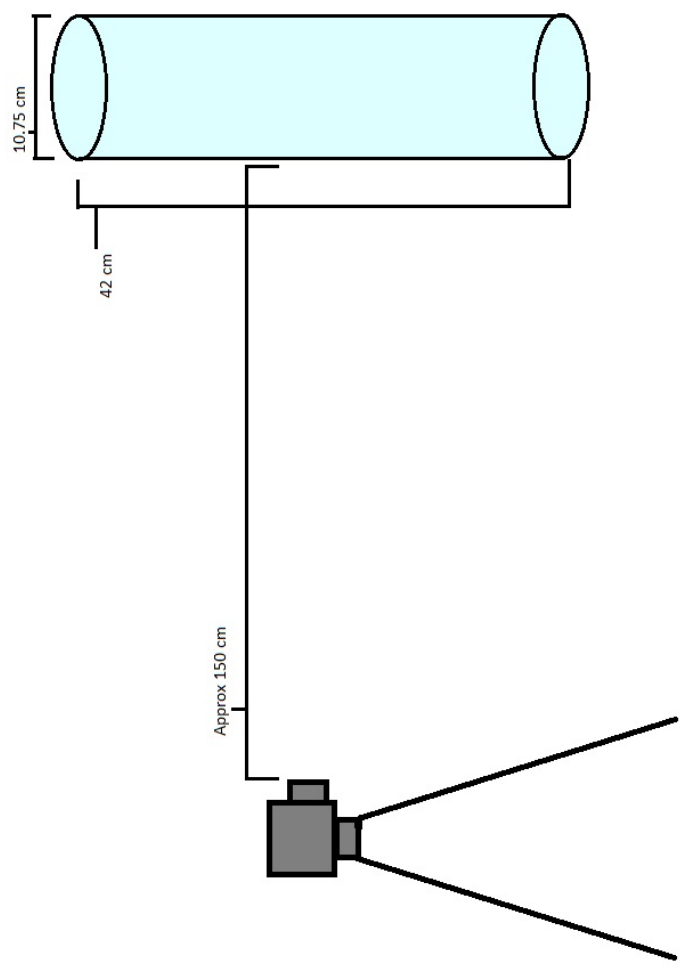



Figure 3: The setup at its most basic level

**Analysis**

This is just a brief note on how analysis was conducted. To extract data from the videos, I used a program for data processing (especially videos) called DataTank. Developed in large by a UNC professor, and available on Apple iOS, DataTank was used for constructing all velocity profiles that you will see throughout this paper. The method of tracking used depended on the experiment, but was predominately through a technique called "color distance". Basically, the program takes a single frame of the video and assign a numeric value to each pixels color. By specifying a specific color and tolerance, DataTank would be able to differentiate colors within that tolerance from the rest of the frame. After finding those pixels within the tolerance, and as long as "enough" are in one spot, the program finds the "center" of these pixels and records its location. As the video evolves, the color moves with the object I am tracking, and thus new locations are recorded at different times. Using this location value and time, it creates a discrete velocity profile from which, using familiar computational methods, creates a continuous velocity profile from which I gather my data.

For some visualizations and computationally rigorous calculations, I differed to the software program Wolfram Mathematica 9. Most useful code developed in Mathematica is included in the end of this paper.

Figure 4: A picture from experiment 9 for better visualization

# 11 Experiment One: Two Identical Spheres in Homogeneous Water

**Date:10/6/2015**

**Problem and Purpose:** To begin my investigation, I started with the simplest case that would still exhibit the behavior we were after, dropping two identical spheres through a homogeneous solution of salt water. This experiment would provide me the opportunity to become familiar with lab protocol, the equipment and material available to me in the lab, and the process of performing experiments of this type. This experiment will also confirm some basic equations and most importantly, it will guide further development of this paper.

**Materials:**
-D7000 Camera (Video Mode Recording)
-Large Plastic Cylinder Tank with Diameter of Approximately 19.2 cm
-Salt Water of Density 1.1774 $g/cm^3$, Assumed viscosity $(mPa.s) \approx 1$
-Ruler (12 inch)
-Giant Plastic Spoon for Retrieval
-Camera Mount
-Sphere Red, Radius(cm)=0.296545, Density$(g/cm^3) \approx 2.26$
-Sphere Black, Radius(cm)=0.296545, Density$(g/cm^3) \approx 2.26$

**Set-up:**The experiment was performed by first filling up the cylindrical tank with the saltwater from the lab's reservoir tank. The density of the water was then checked with the portable density meter. The tank was moved to the table where I then placed a grided backdrop. Using adhesive putty, I stuck a ruler to the side of the tank so that the side with measurement notches would be facing the camera. I then set up the camera on the stand, focused it on the ruler (the plane where the sphere would be dropped) and stabilized at an appropriate height. Lastly, I set up the lighting apparatus on a rig behind the camera.

**Procedure:** The experiment was performed in the following manner. First there were three runs of dropping both the black and the red sphere simultaneously, tangentially and side-by-side. Then, I performed alternating drops of a single red then black sphere, so as to have three videos for each situation.

**Results:**

| Type | Terminal Velocity* | Error |
|---|---|---|
| Single Sphere Predicted (Morrison's Coeff.)) | 39.5845 | NA |
| Single Sphere (Red) | 41.93188 | 5.93% |
| Single Sphere (Black) | 39.63765 | .134% |
| Double Sphere (Horizontal)(Red) | 40.74412 | NA |
| Double Sphere (Horizontal)(Black) | 41.33375 | NA |

* Calculated by average of the maximum velocity from tracking and the two velocity points beside it.

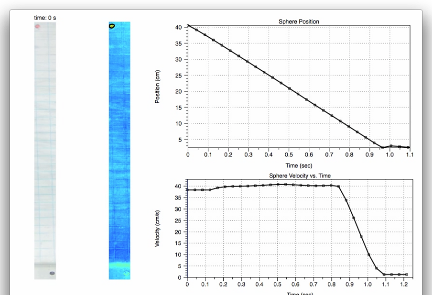



Figure 5: Single Sphere (Red)

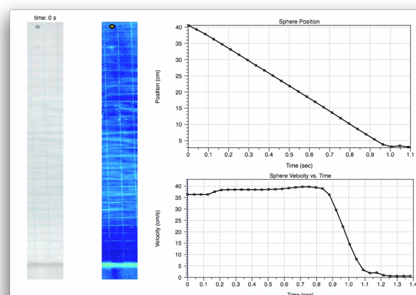



Figure 6: Single Sphere (Black)

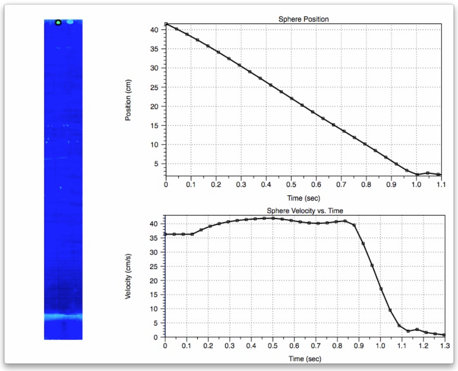



Figure 7: Double Sphere (Horizontal) (Red)

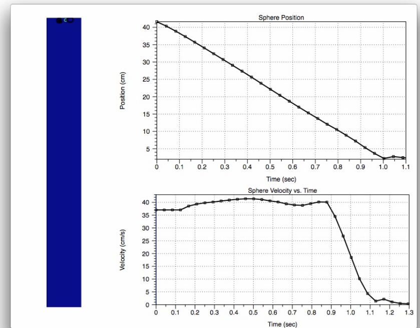



Figure 8: Double Sphere (Horizontal)(Black)

**Issues:** As observed by the lack of data, the lighting needs to be improved as to minimize shadows, so that more positions can be tracked. It should also be mentioned that a sinusoidal falling pattern created a poor data set of the "true" velocity since DataTank was only tracking the vertical descent (would have been higher). As recommend by Doctor Camassa and Graduate Claudia Falcon, a more viscous fluid will prevent this behavior, creating a less deviated linear path from top to bottom by minimizing/removing the "vortex rings" that it creates on its descent. Likewise, the more vicious substance will allow for more data points as the sphere takes longer to fall from top to bottom. Also, the camera was not focused enough and adjustments could be made to the aperture and shutter speed.

**Conclusion:** The experiment was for a large part a success and seems to agree with our prediction that two spheres do indeed fall faster then one. Now we need to move from finite Reynolds to low Reynolds corn syrup to remove the sinusoidal behavior from the vortex ring, and to provide enough tracking time to have sufficient data points. It also seems that the two spheres are reasonably identical. Future tests will reaffirm or deny this.

# 12 Experiment Two: Two Identical Spheres in Homogeneous Corn Syrup

**Date:10/12/2015**

**Problem and Purpose:** So now that I had some experience in the lab and some clear evidence of two spheres falling faster then one, I followed the recommendation of my adviser, and transitioned to corn syrup, where the dampening of the sinusoidal behavior, the low Reynolds domain, and presence of more data will allow for a more accurate representation of this effect. Likewise, I will be performing multiple analysis on each configuration, experimenting with different methods of releasing the sphere and trying new configurations. This shall be the precursor to moving to a stratified regime and the actual behavior we are trying to observe.

**Materials:**
-D7000 Camera (Manual Mode Button Press Interval Shooting)
-Small Plastic Cylinder Tank with Diameter of Approximately 10.75cm
-Corn Syrup of Density of Approximately 1.36672 $g/cm^3$ at 22.65 $C$ (Density, Viscosity and Temperature was measured four days after experiment), Viscosity 22.1281 $Pa.s$
-Camera Mount
-Plastic Wrap
-Metal Mixing Pot
-Giant Plastic Spoon for retrieval and stirring
-Sphere Red, Radius(cm)=0.296545, Density($g/cm^3$) $\approx$ 2.26
-Sphere Black, Radius(cm)=0.296545, Density($g/cm^3$) $\approx$ 2.26

**Set-up:**The tank was first washed. Then, a pot was filled with corn syrup directly from the sealed containers. It was then mixed by hand for 30 minutes to create homogeneity. Next, the corn syrup was poured into the cylindrical tank, covered in plastic wrap to prevent evaporation and put aside to de-gas overnight. The lighting apparatus was placed behind the camera. The camera was stabilized and set to optimum aperture and shutter speed. The tank was moved to the optimum distance away from the camera and then a backdrop grid placed. I attached a ruler to the side of the tank with putty, so the face would be directly facing the camera. Lastly, I focused the camera on the ruler (the plane that the sphere should fall in).

**Procedure:** The experiment was performed in the following manner. First I alternately dropped a single black sphere, then a single red. This, pattern was repeated three times, however, only two videos of the black sphere and one of the red where suitable for keeping after later analysis. I then conducted three drops with the "Snowman" configuration of vertical aligned tangential spheres, only one of which were appropriated for analysis. Lastly, I performed four videos of a vertical alignment, only two of which were suitable for analysis. After each drop, the spheres were removed from the bottom using the giant plastic spoon, partially submerging my hand, then quickly cleaning the apparatus and the sphere. The camera was set to a manual mode, where it would take 1 second photos as long as I held the button. After dropping the spheres I would immediately press the button and wait till they reached the bottom of the tank to release.

**Results:**

| Type | Terminal Velocity | Average | Prediction Error |
|---|---|---|---|
| Single Sphere Predicted | 0.740808 | NA | NA |
| Single Sphere (Red) | 1.060371 | 1.00791 | 43.13 % |
| Single Sphere (Black)(1) | 1.022109 | 1.00605 | 38.00% |
| Single Sphere (Black)(2) | 1.023222 | .964319 | 38.12 % |
| Double Sphere (Horizontal) (Predicted) | 1.06491 | NA | NA |
| Double Sphere (Horizontal) (Red) | 1.358423 | 1.3337 | 27.56% |
| Double Sphere (Horizontal) (Black) | 1.377877 | 1.34705 | 29.39% |
| Double Sphere (Horizontal) (Red)(2) | 1.420145 | 1.3763 | 33.36% |
| Double Sphere (Horizontal) (Black)(2) | 1.410569 | 1.37522 | 32.46% |
| Double Sphere (Vertical) Predicted | 1.20381 | NA | NA |
| Double Sphere (Vertical*)(Red) | 1.551565 | 1.53159 | 28.89% |
| Double Sphere (Vertical*)(Black) | 1.529734 | 1.49939 | 27.07% |

*Red-on-Black

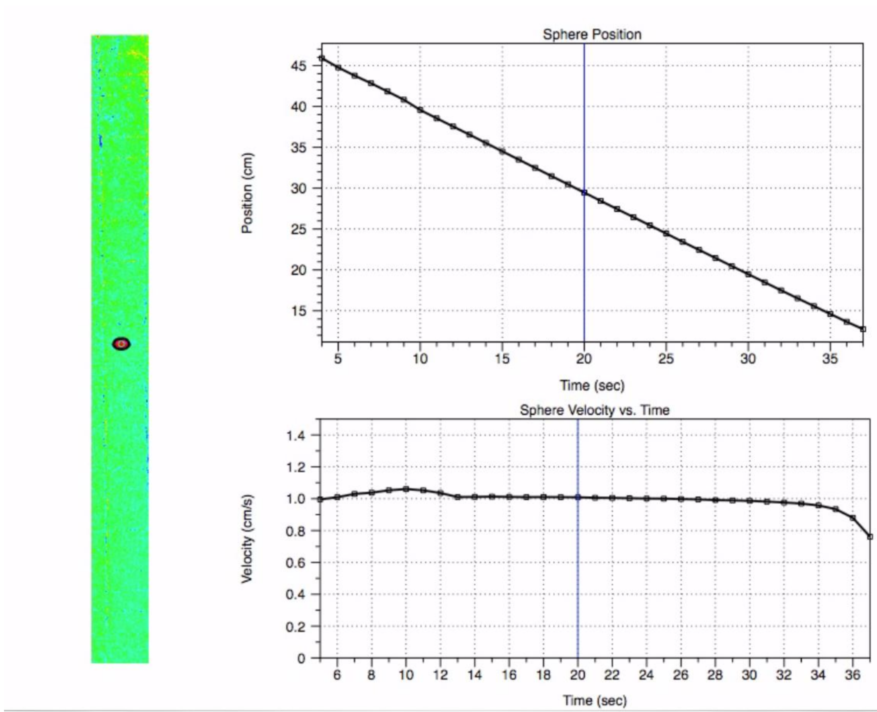



Figure 9: Single Sphere (Red)

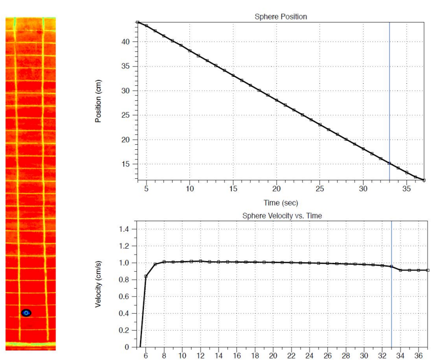


Figure 10: Single Sphere (Black)

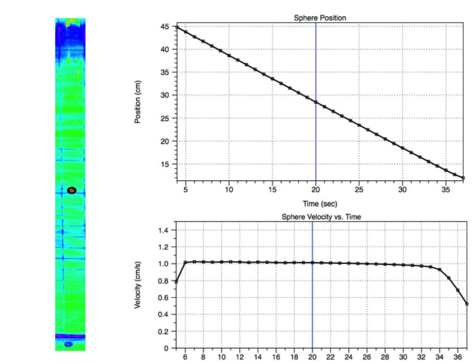


Figure 11: Single Sphere (Black)(2)

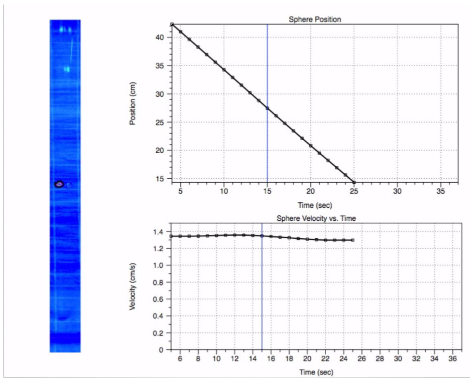



Figure 12: Double Sphere (Horizontal) (Red)

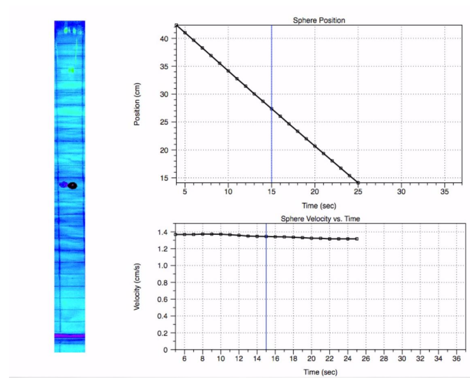



Figure 13: Double Sphere (Horizontal) (Black)

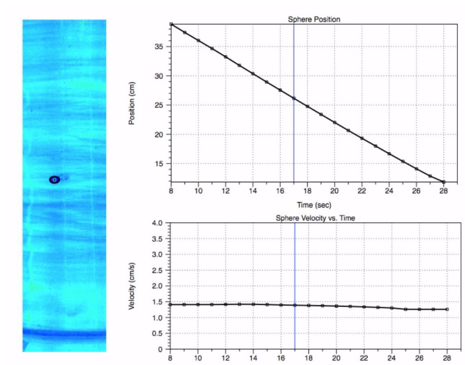



Figure 14: Double Sphere (Horizontal) (Red)(2)

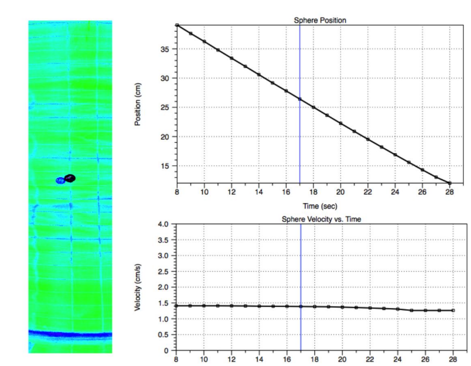



Figure 15: Double Sphere (Horizontal) (Black)(2)

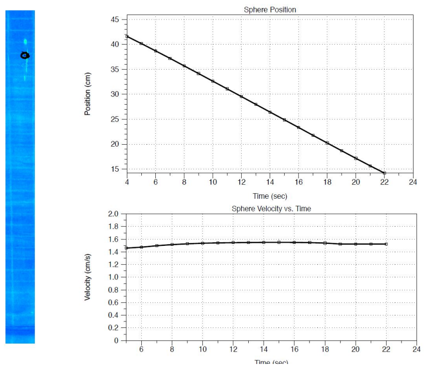


Figure 16: Double Sphere (Vertical/Red-on-Black)(Red)

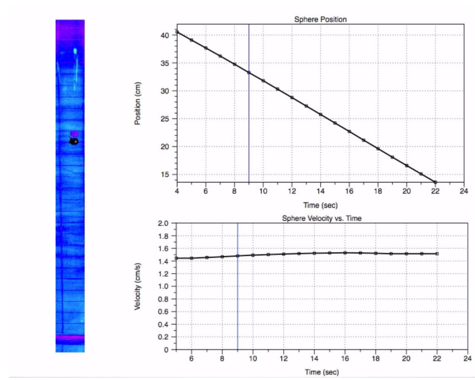


Figure 17: Double Sphere (Vertical/Red-on-Black)(Black)

**Issues:** The cameras lighting looked good on digital display while recording, but during analysis it became apparent it could have been brighter. Shadows from the top of the tank and the reflection of the light prevented good tracking for the first part of the drop. Likewise, the bottom part of the drop remained unanalyzed due to some finicky lighting behavior toward the bottom. The markers coloration on the spheres was slowly eroded as tests where performed, and although I recolored it twice during the experiments, it would have been much easier to track with a darker, even coat that didn't remove itself. When I would retrieve the sphere with the spoon, I would introduce bubbles and get my hand extremely sticky. Also, on a couple occasions after cleaning the spoon in a nearby wash pot, I introduced water into the upper levels of the corn syrup. Although I settled on my fingers, the funnel and two pieces of cardboard (acted as a vice) that I used for dropping, caused very horrendous looking drops that ultimately could not be used. Even still, my fingers were very much imperfect and the boundary layer they created caused a uneven drop. Also, since I had to hold the camera button down while shooting, I couldn't replace the plastic wrap as the video ran, meaning the top part of the corn syrup was evaporating and becoming more dense then it should have. Lastly, there should have been more pictures taken and used. There was simply not enough data points, that covered to little of the range of the corn syrup in the tank, and not enough videos of each method of dropping.

**Conclusion:** The experiment was in some respects, a success. We eliminated the sinusoidal falling behavior as predicted, got more data points, and produced precise and consistent plots and data. However, we have a huge discrepancy in our theoretically predicted terminal velocity and our experimentally observed. As noted by Claudia Falcon, the camera is inconsistent with interval shooting, so we will be using video mode next time. Likewise, we will be taking viscosity the day of.

# 13 Error Propagation

Ultimately, the tools that I am using to perform measurements are inaccurate to some mechanical degree, and small deviations from the true measurements can grow exponentially while using the high order and many termed equations that I am using for predicting terminal velocity of a single particle. As of such, it is of utmost importance that we look at how these errors can propagate and contribute for each function for our Coefficient of Drag: Stoke's, Oseen's, and Morrison's.

**Errors from Each Tool**

Scale error = $5 * 10^{-4}$ grams as stated by manufacturer's website
Caliper error = .0001 inches from observation
Density meter error = .00001$g/cm^3$ as stated by manufacturer's website
Viscometer error = .5 percent of dynamic viscosity as stated by manufacturer's website

**How that Showed in Experiment Two**

Mass = .247033 (Measured Mass in grams)
Radius = .2975991 (Converted Measured Radius in cm)
Fluid Density= 1.36672 (Measured Density in $g/cm^3$)
Dynamic Viscosity = 21.13Pascal seconds (Measured Dynamic Viscosity)

Using a range of values for possible error where the magnitude was less then or equal to the maximum error and solving the respective equation with those numbers gave us a 4-dimensional matrix of terminal velocities. The maximum element in the matrix is the biggest terminal velocity that can be achieved with that error in mind.

**Stoke's**

This is the easiest to start with as the $C_d = 24/Re$ simplifies equation 1 dramatically, to equation 2.

With Measured Values:$V_{Terminal} = 0.795702$

With Maximum Error:$V_{Terminal} = 0.801771$
At values:
-Mass = 0.247533 grams
-Radius= 0.2974721 centimeters
-Density of fluid= 1.33671 $g/cm^3$
-Dynamic Viscosity of Liquid = 21.13 pascal seconds

**Oseen's**

With Measured Values:$V_{Terminal} = 0.791183$

With Maximum Error:$V_{Terminal} = 0.798907$
At values:
-Mass = 0.247533 grams
-Radius= 0.2974721 centimeters
-Density of fluid= 1.33671 $g/cm^3$
-Dynamic Viscosity of Liquid = 21.13 pascal seconds

**Morrison's**

With Measured Values:$V_{Terminal} = 0.795526$

With Maximum Error:$V_{Terminal} = 0.801685$

At values:
-Mass = 0.247533 grams
-Radius= 0.2974721 centimeters
-Density of fluid= 1.33671 $g/cm^3$
-Dynamic Viscosity of Liquid = 21.13 pascal seconds

**Conclusion:** So it seems that instrument error can't account for the massive discrepancy. Although this was calculated for a particular experiment, this could have readily been generalized. Whats important is that the order of this error will remain constant across the experiments of this paper and thus, negligible compared to the error we are seeing. This leaves two possibility in my mind, either I messed up as an experimenter, or the camera was inaccurately taking interval shots (suggested by mentor Claudia Falcon). Either way, a retest is in order that will attempt to anticipate and eliminate these possibilities.

# 14 Experiment Three: Re-Test Two Identical Spheres in Homogeneous Corn Syrup

**Date:11/11/2015**

**Problem and Purpose:** In an attempt to find the source of the discrepancy between the calculated and the experimental terminal velocity, a re-test was needed. To eliminate some errors: this test will be performed by multiple experimenters and tracked with video shooting to get more frames, all density and samples will be collected the day of the experiment, we will be concentrating on replacing the plastic wrap covering between test to prevent evaporation, and we will attempt to reduce foreign liquids and air bubbles from entering the corn syrup. This test will have a focus on single drop and double vertical drop tests with different distances to provide a stronger transition to a two layered regime.

**Materials:**
-D7000 Camera (Video Mode Recording)
-Small Plastic Cylinder Tank with Diameter of Approximately 10.75cm
-Corn Syrup of Density 1.36902 $g/cm^3$ at 22.90 $C$ with a Dynamic Viscosity of 23.632 Pascal Seconds
-Camera Mount
-Plastic Wrap
-Metal Mixing Pot
-Small Standard Plastic Spoon Attached to Metal Rod

-Sphere Red, Radius(cm)=0.296545, Density($g/cm^3$) $\approx$ 2.26
-Sphere Black, Radius(cm)=0.296545, Density($g/cm^3$) $\approx$ 2.26

**Set-up:** Corn syrup from Experiment Two was poured into the metal mixing pot and a small amount of additional corn syrup was added from the stored containers. It was then stirred for 40 minutes, poured back into the cylinder, covered in two layers of plastic wrap to prevent evaporation, and left over two nights. The day of the experiment (with the assistance of Gabbi and Claudia Falcon), the density was taken, as well as a sample for the viscometer. The camera was mounted, balanced and focused. Lastly, paper towels were procured from the bathrooms and the sphere were recolored with marker.

**Procedure:** The experiment began with a single horizontal drop, after this, there were three drops with the red bead and three with the black. Lastly, there were four vertical drops, three of which were red on black with alternate distances and one black on red. As in previous experiments, the sphere(s) would be lowered just below the interface by hand, rolled to remove bubbles, dropped, and the we slowly moved our hands away as to minimize disturbance to the configuration or velocity. We then retrieved the sphere with the plastic spoon, being careful to minimized disturbance to the interface. Between experiments, we covered the top of the tank with plastic wrap, cleaned the retrieval tool, the spheres, the tank and recolored the spheres.
**Results:**

| Type | Terminal Velocity | Predicted | Error |
|---|---|---|---|
| Single Sphere (Red)(1) | 0.795560 | 0.639114 | 24.47% |
| Single Sphere (Black)(1) | 0.826212 | 0.639114 | 29.27% |
| Single Sphere (Red)(2) | 0.807280 | 0.639114 | 26.31% |
| Single Sphere (Black)(2) | 0.808012 | 0.639114 | 26.42% |
| Single Sphere (Red)(3) | 0.794257 | 0.639114 | 24.27% |
| Single Sphere (Black)(3) | 0.826110 | 0.639114 | 29.26% |
| Double Sphere (Vertical/Red-on-Black/0.10cm)(Red)(1) | 1.208318 | 0.999232 | 20.92% |
| Double Sphere (Vertical/Red-on-Black/0.10cm)(Black)(1) | 1.194725 | 0.999232 | 19.56% |
| Double Sphere (Vertical/Red-on-Black/2.75cm)(Red)(2) | 0.875607 | 0.723706 | 20.99% |
| Double Sphere (Vertical/Red-on-Black/2.75cm)(Black)(2) | 0.858294 | 0.723706 | 18.60% |
| Double Sphere (Vertical/Red-on-Black/0.37cm)(Red)(3) | 1.126024 | 0.915641 | 22.98% |
| Double Sphere (Vertical/Red-on-Black/0.37cm)(Black)(3) | 1.107042 | 0.915641 | 20.90% |
| Double Sphere (Vertical/Black-on-Red/0.87cm)(Red)(4) | 1.019809 | 0.8281 | 23.15% |
| Double Sphere (Vertical/Black-on-Red/0.87cm)(Black)(4) | 1.046783 | 0.8281 | 26.41% |
| Double Sphere (Vertical/Black-on-Red/4.02cm)(Red)(4) | 0.826242 | 0.700571 | 17.94% |
| Double Sphere (Vertical/Black-on-Red/4.02cm)(Black)(4) | 0.829505 | 0.700571 | 18.40% |

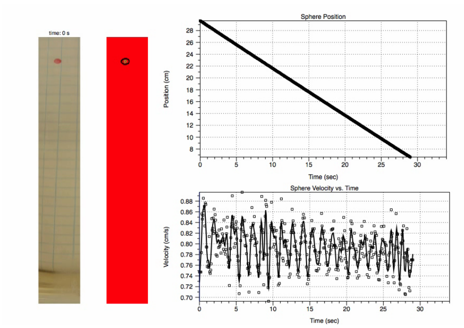


Figure 18: Single Sphere (Red)(1)

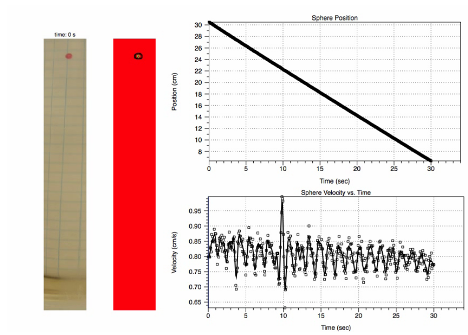


Figure 19: Single Sphere (Red)(2)

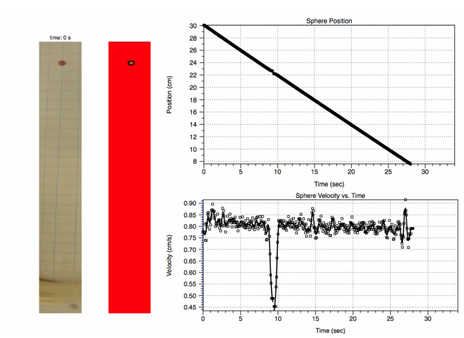


Figure 20: Single Sphere (Red)(3)

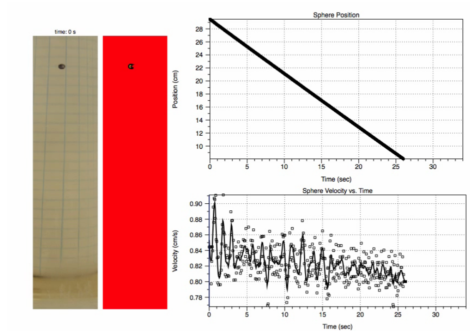


Figure 21: Single Sphere (Black)(1)

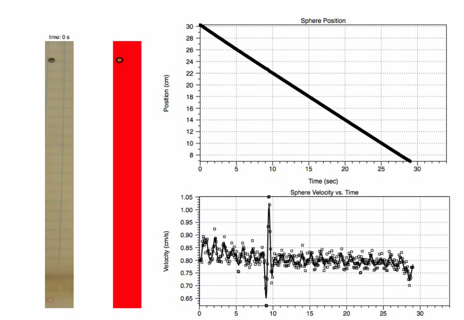


Figure 22: Single Sphere (Black)(2)

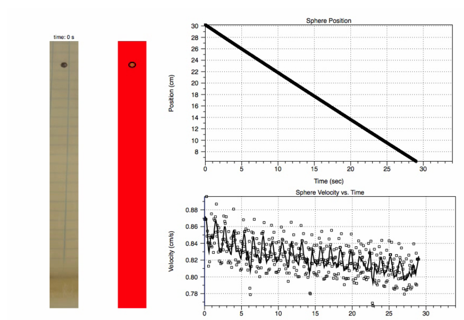


Figure 23: Single Sphere (Black)(3)

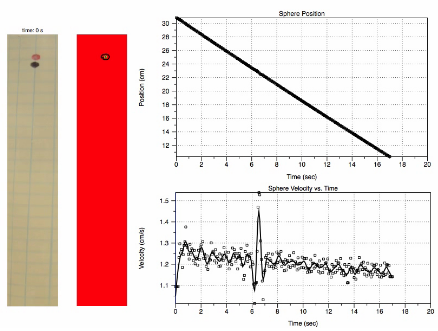



Figure 24: (Vertical/Red-on-Black/0.10cm)(Red)(1)

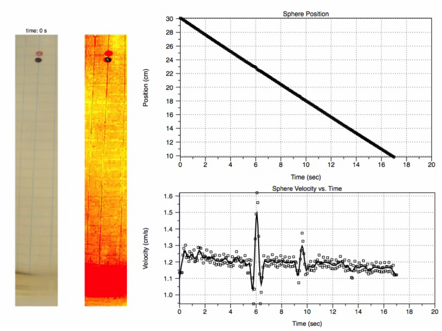



Figure 25: (Vertical/Red-on-Black/0.10cm)(Black)(1)

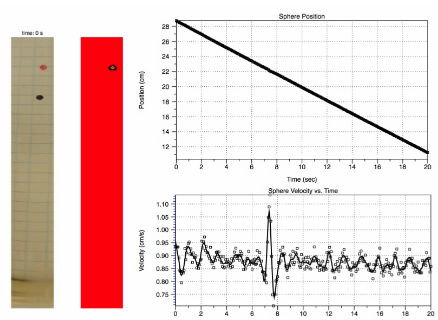



Figure 26: (Vertical/Red-on-Black/2.75cm)(Red)(2)

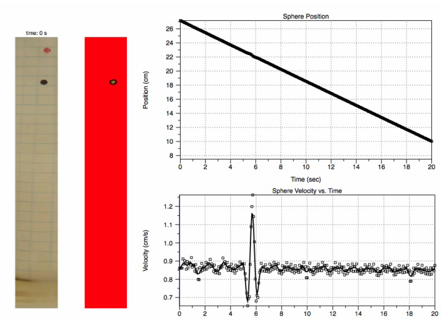



Figure 27: (Vertical/Red-on-Black/2.75cm)(Black)(2)

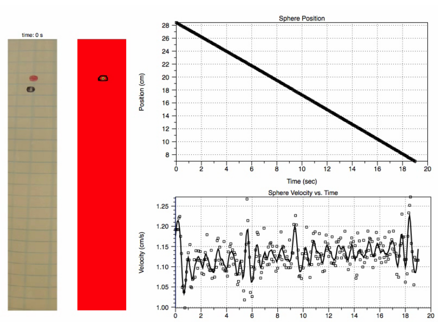



Figure 28: (Vertical/Red-on-Black/0.37cm)(Red)(3)

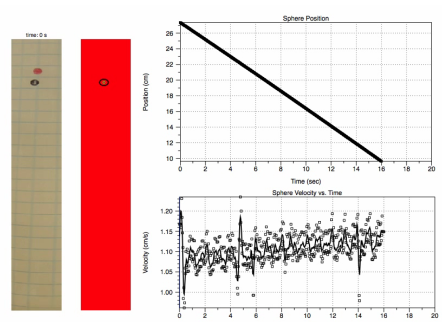



Figure 29: (Vertical/Red-on-Black/0.37cm)(Black)(3)

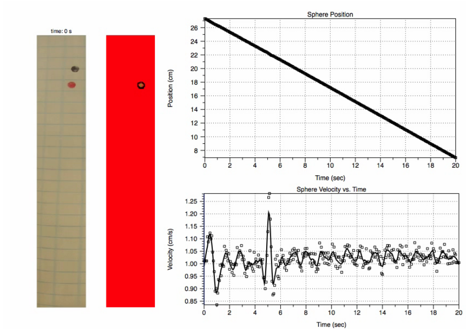



Figure 30: (Vertical/Black-on-Red/0.87cm)(Red)(4)

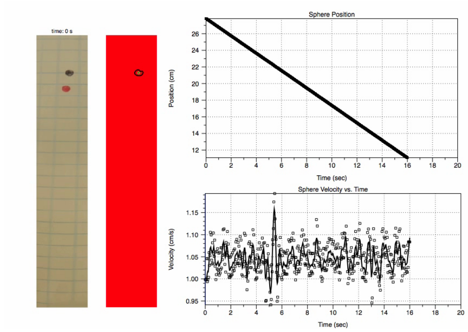



Figure 31: (Vertical/Black-on-Red/0.87cm)(Black)(4)

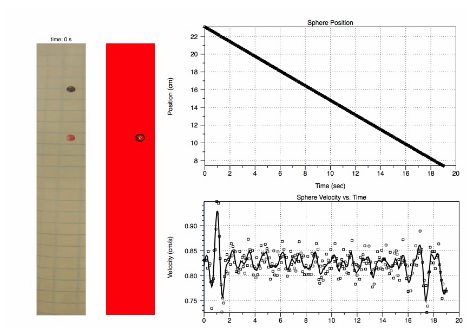



Figure 32: (Vertical/Black-on-Red/4.02cm)(Red)(4)

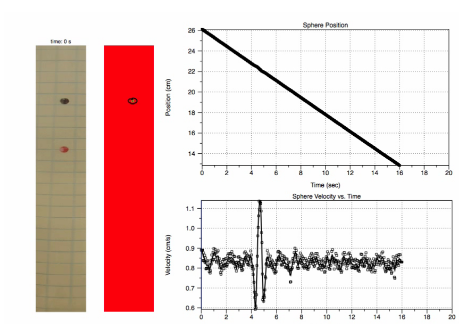



Figure 33: (Vertical/Black-on-Red/4.02cm)(Black)(4)

**Issues:** The focus should had been checked between test. By coloring over the already dried marker on the spheres, the old marker coloration became wet and easily came off during experiments. After each run, the sphere would have to be cleaned and recolored, making for very poor coloring overall. In most of the tracking, a glob of corn syrup on the outside of the tank messed up the tracking for several frames creating dramatic jumps in the velocity tracking and the graph. This implies that the tank should be more thoroughly cleaned between experiments. The vertical drops were near impossible to get to drop tangentially. Likewise, for other vertical drops, the distance couldn't be controlled for and was calculated post experiment. The retrieving mechanism had a crevice between the spoon, the rod and the duct tape that held the two together. This led to a constant stream of giant bubbles entering the corn syrup. Like usual, the plastic wrap was hard to get back on without affecting the initial fall of the sphere. More paper towels were needed.

**Conclusion:** The experiment succeeded in reasonably eliminating the two assumed possibilities of error that the experiment was designed to eliminate: human error and time measurement issues with interval shooting, but all the same, still produced the frightening discrepancy in experimental versus model. In other words, the error is neither from the measurement tools (unless they are broken or not calibrated correctly) nor from the experimenters, indicating that we need to look into other possibilities.

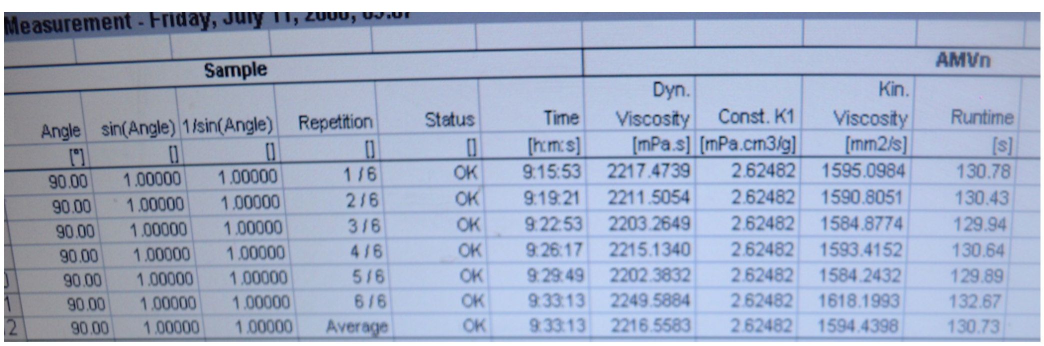
# 15 Issues with the Viscometer and Alternatives

### Viscometer Issues

As noted in the experimental conclusions, we are getting a large discrepancy between predicted terminal velocity and experimentally observed values:

A large part of my research up to this point has been put into finding its source. First, we thought that there was an issue with Morrison's equation, which worked for the saltwater, but had issues at low Reynolds with the corn syrup. This led to the exploration into its behavior around zero (Taylor Expansion) and me switching to Stokes drag. After finding out that the equation wasn't an issue, we looked into instrumental error. After concluding that even that was not substantial enough to account for the error, we thought it may have been experimental [note all the issues in experiment two]. My prediction was that the cameras picture timer was off and wasn't accurately taking one second interval photos or that I had messed up as an experimenter. So in consequence, experiment three was ran to compensate for this and other possible issues by having multiple experimenters present, and using video recording instead of interval photos. Yet still we had discrepancy between the predicted and the experimentally observed. Finally, after beginning a retesting of the devices for error, it struck me how strange it was that there was such a large deviation between even and odd measurement by the viscometer. To elaborate, the viscometer runs multiple measurements by rotating a glass capillary with a gold bead suspended in the to be measured fluid, then tracks the beads descent using electro-conductivity. Using this data, and a standard of calibration, and a density it can measure viscosity. thus measurements are formed in pairings with the starting orientation being the odd measurements (the 1st, 3rd, etc.), and the flipped orientation being the even. So after a quick look into previous data charts, it became apparent that this large deviation of up to 9 Pascal Seconds was non-existent prior to two measurements ran earlier before my measurements where recorded.

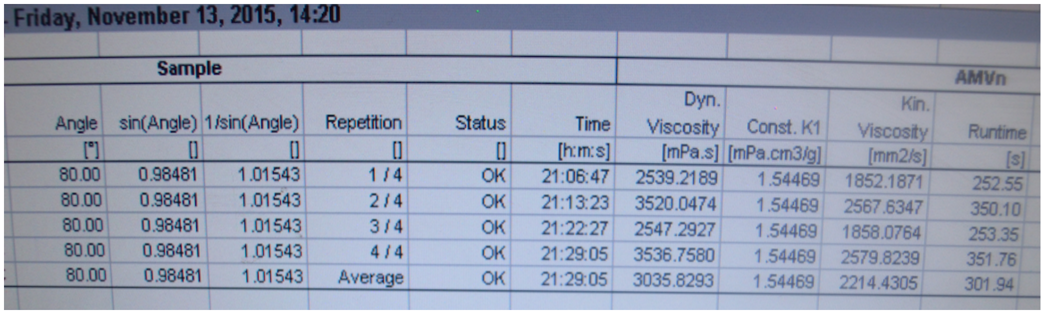
**From 2008:**



Measurement - Friday, July 11, 2008

| Sample | | | | | | AMVn | | | |
|---|---|---|---|---|---|---|---|---|---|
| Angle [°] | sin(Angle) [] | 1/sin(Angle) [] | Repetition [] | Status [] | Time [h:m:s] | Dyn. Viscosity [mPa.s] | Const. K1 [mPa.cm3/g] | Kin. Viscosity [mm2/s] | Runtime [s] |
| 90.00 | 1.00000 | 1.00000 | 1 / 6 | OK | 9:15:53 | 2217.4739 | 2.62482 | 1595.0984 | 130.78 |
| 90.00 | 1.00000 | 1.00000 | 2 / 6 | OK | 9:19:21 | 2211.5054 | 2.62482 | 1590.8051 | 130.43 |
| 90.00 | 1.00000 | 1.00000 | 3 / 6 | OK | 9:22:53 | 2203.2649 | 2.62482 | 1584.8774 | 129.94 |
| 90.00 | 1.00000 | 1.00000 | 4 / 6 | OK | 9:26:17 | 2215.1340 | 2.62482 | 1593.4152 | 130.64 |
| 90.00 | 1.00000 | 1.00000 | 5 / 6 | OK | 9:29:49 | 2202.3832 | 2.62482 | 1584.2432 | 129.89 |
| 90.00 | 1.00000 | 1.00000 | 6 / 6 | OK | 9:33:13 | 2249.5884 | 2.62482 | 1618.1993 | 132.67 |
| 90.00 | 1.00000 | 1.00000 | Average | OK | 9:33:13 | 2216.5583 | 2.62482 | 1594.4398 | 130.73 |

**From 2015:**



Friday, November 13, 2015, 14:20

| Sample | | | | | | | | | AMVn |
|---|---|---|---|---|---|---|---|---|---|
| Angle | sin(Angle) | 1/sin(Angle) | Repetition | Status | Time | Dyn. Viscosity | Const. K1 | Kin. Viscosity | Runtime |
| [°] | [] | [] | [] | [] | [h:m:s] | [mPa.s] | [mPa.cm3/g] | [mm2/s] | [s] |
| 80.00 | 0.98481 | 1.01543 | 1 / 4 | OK | 21:06:47 | 2539.2189 | 1.54469 | 1852.1871 | 252.55 |
| 80.00 | 0.98481 | 1.01543 | 2 / 4 | OK | 21:13:23 | 3520.0474 | 1.54469 | 2567.6347 | 350.10 |
| 80.00 | 0.98481 | 1.01543 | 3 / 4 | OK | 21:22:27 | 2547.2927 | 1.54469 | 1858.0764 | 253.35 |
| 80.00 | 0.98481 | 1.01543 | 4 / 4 | OK | 21:29:05 | 3536.7580 | 1.54469 | 2579.8239 | 351.76 |
| 80.00 | 0.98481 | 1.01543 | Average | OK | 21:29:05 | 3035.8293 | 1.54469 | 2214.4305 | 301.94 |

**Alternative: Viscosity Cups**

While we attempted to fix the viscometer, we needed to find an alternative way to measure the viscosity for experiment three before the corn syrup in the tank begins to evaporate and change viscosity. Adviser Camassa recommended that I try using "Viscosity Cups". These are metal cups with different volumes and sized holes in the bottom for different ranges of velocities. After filling them up with the liquid you want to measure, your record the amount of time it takes for the fluid to run out (for the falling stream to have a definite break). Based off of the time and a table, you can retrieve a viscosity.

**11/25/2015**

Averaging 5 separate measurements from viscosity cup number 5, we averaged at 60.13 seconds. The table gave us a measurement of 1401 centiStokes, which with a density of 1.36902 $g/cm^3$, gives us 19.18 Pascal Seconds. This gives us the new table:

**Results:**

| Type | Terminal Velocity | Predicted | Error |
|---|---|---|---|
| Single Sphere (Red)(1) | 0.795560 | 0.787463 | 1.03% |
| Single Sphere (Black)(1) | 0.826212 | 0.787463 | 4.92% |
| Single Sphere (Red)(2) | 0.807280 | 0.787463 | 2.52% |
| Single Sphere (Black)(2) | 0.808012 | 0.787463 | 2.61% |
| Single Sphere (Red)(3) | 0.794257 | 0.787463 | 0.86% |
| Single Sphere (Black)(3) | 0.826110 | 0.787463 | 4.91% |
| Double Sphere (Vertical) (Red-on-Black) 0.10cm(Red)(1) | 1.208318 | 1.23117 | 11.58% |
| Double Sphere (Vertical) (Red-on-Black) 0.10cm)(Black)(1) | 1.194725 | 1.23117 | 10.33% |
| Double Sphere (Vertical) (Red-on-Black) 2.75cm)(Red)(2) | 0.875607 | 0.89169 | 8.45% |
| Double Sphere (Vertical) (Red-on-Black) 2.75cm)(Black)(2) | 0.858294 | 0.89169 | 6.31% |
| Double Sphere (Vertical) (Red-on-Black) 0.37cm)(Red)(3) | 1.126024 | 1.12818 | 12.68% |
| Double Sphere (Vertical) (Red-on-Black) 0.37cm)(Black)(3) | 1.107042 | 1.12818 | 10.78% |
| Double Sphere (Vertical) (Red-on-Black) 0.87cm)(Red)(4) | 1.019809 | 1.02032 | 11.85% |
| Double Sphere (Vertical) (Red-on-Black) 0.87cm)(Black)(4) | 1.046783 | 1.02032 | 14.81% |
| Double Sphere (Vertical) (Red-on-Black) 4.02cm)(Red)(4) | 0.826242 | .863185 | 5.36% |
| Double Sphere (Vertical) (Red-on-Black) 4.02cm)(Black)(4) | 0.829505 | .863185 | 5.77% |

Amazingly, considering the 12 day separation, these numbers are remarkably close to what we predicted. Providing more evidence toward a faulty viscometer.

**Fixing the Viscometer:**

After a couple week span of calling Anton Paar, being told we had a bad belt and needed to send it in for repairs, attempting to schedule with other universities to minimize travel cost for repairman, then lastly fixing the viscometer with the assistance of a technician over phone, we were finally able to both confirm that the viscometer was providing false reading and get it in working order.

# 16 Dropping Apparatus

To control drop separation and provide precise positioning for different configurations I have developed a dropping apparatus:

**Objects:**
-Disassembled mechanical pencil green plastic casing
-Two Doweling Rods
-Insulating Tape
-Hot Glue

**Pros:** Is a great improvement on retrieval (as opposed to the spoon), punctures the evaporated surface interface without tainting the sphere(s), drops very consistently, is easier to manipulate orientation and separation of spheres, less of a mess for hands.

**Cons:** Has to be cleaned after each run, requires a separate holding beaker of top layer, it is hard not to tilt tool when dropping (no structure to keep everything aligned in tank).

(Note:Is approximately a three feet long)

# 17 Experiment Four: Second Re-Test of Two Identical Spheres in Homogeneous Corn Syrup

**Date:12/15/2015**

**Problem and Purpose:** Although experiment three looked incredible after taking the velocity cups into consideration, we still need two things before we make the big step into a stratified regime: more data points, a viscosity taken from an instrument that is both precise and built to measure at the viscosity of our fluid. In other words, this experiment is just a reassurance before we continue into the stratified regime.

**Materials:**

-D7000 Camera (Video Mode Recording)
-Small Plastic Cylinder Tank with Diameter of Approximately 10.75cm
-Corn Syrup of Density of Approximately 1.37440 $g/cm^3$ and Viscosity 24.5190$Pa.s$ Viscometer Cups, and Density of 1.37537 $g/cm^3$ and Viscosity 26.3997$Pa.s$ Viscometer (12/18/2015)
-Camera Mount
-Plastic Wrap
-Metal Mixing Pot
-Giant Plastic Spoon for stirring
-Designed Tool for Retrieval and Dropping
-Sphere Red, Radius(cm)=0.296545, Density($g/cm^3$) $\approx$ 2.26
-Sphere Black, Radius(cm)=0.296545, Density($g/cm^3$) $\approx$ 2.26

**Set-up:** Corn syrup from Experiment Three was poured into the metal mixing pot and additional corn syrup was added from the stored containers. It was then stirred for 40 minutes, poured back into the cylinder, covered in two layers of plastic wrap to prevent evaporation, and left over two nights. The day of the experiment, the spheres were recolored with marker, the density was taken, the viscosity was measured with the viscosity cups (with the help of Claudia Falcon), and a sample for the viscometer (still not working at this point) was taken. The camera was mounted, balanced and focused. A large beaker full of corn syrup from the tank that is used for holding the retrieval tool, priming it for the drop, and holding the spheres was put beside the tank. The netal mixing pot is palced beside the tank, full of water, for cleaning. Lastly, paper towels were procured from the bathrooms. (Note: Viscosity was taken again three days later after viscometer was fixed).

**Procedure:** The experiment consisted of 18 drops, of which, 14 were considered suitable for analysis. An experiment was performed in the following manner, spheres are submerged in holding beaker. Focus and orientation of camera is checked. Record button is pressed. Tool is submerged in holding beaker, and sphere are sucked up. Tool is lifted from beaker and hand cups bottom to prevent dripping on table and tank. Tool is submerged. Orientation of tool in liquid is checked. Spheres are slowly pushed out and distance adjusted. Slowly tool is removed and I back away as to prevent shadows. I wait for spheres to stop falling, then Stop the recording. Retrieve spheres using tool. Drop spheres in metal pot full of water. Clean tool, then dry tool. Clean spheres under water, then dry spheres. Put spheres into holding beaker. Repeat.

**Results:**

| Type | Distance | Terminal Velocity | Stand. Dev. from Terminal | Predicted | Error |
|---|---|---|---|---|---|
| Single Sphere (Red)(1) | NA | 0.5763203 | 0.03101 | 0.571759 | .7915 % |
| Single Sphere (Black)(1) | NA | 0.578095 | 0.00904 | 0.571759 | 1.09601% |
| Verticle Double Sphere BR(Red)(1) | 0.069428 | 0.8397748 | 0.01663 | 0.904367 | 7.69% |
| Verticle Double Sphere BR(Black)(1) | 0.069428 | 0.8365177 | 0.01884 | 0.904367 | 8.11% |
| Verticle Double Sphere (Red)(2) | 0.0971035 | 0.821565 | 0.02923 | 0.894898 | 8.93% |
| Verticle Double Sphere (Black)(2) | 0.0971035 | 0.8176483 | 0.02268 | 0.894898 | 9.45% |
| Verticle Double Sphere (Red)(3) | 0.236437 | 0.8101529 | 0.0260 | 0.852232 | 5.19% |
| Verticle Double Sphere (Black)(3) | 0.236437 | 0.8063606 | 0.01942 | 0.852232 | 5.69% |
| Verticle Double Sphere (Red)(4) | 0.258821 | 0.7655826 | 0.03126 | 0.846182 | 10.53% |
| Verticle Double Sphere (Black)(4) | 0.258821 | 0.7649389 | 0.02393 | 0.846182 | 10.62% |
| Verticle Double Sphere (Red)(5) | 0.269877 | 0.7859402 | 0.02851 | 0.843272 | 7.29% |
| Verticle Double Sphere (Black)(5) | 0.269877 | 0.7812826 | 0.02529 | 0.843272 | 7.93% |
| Verticle Double Sphere (Red)(6) | 0.40301 | 0.7037796 | 0.02845 | 0.811997 | 15.38% |
| Verticle Double Sphere (Black)(6) | 0.40301 | 0.700383 | 0.01794 | 0.811997 | 15.94% |
| Verticle Double Sphere (Red)(7) | 0.841339 | 0.6991079 | 0.02829 | 0.74401 | 6.42% |
| Verticle Double Sphere (Black)(7) | 0.841339 | 0.6976924 | 0.02093 | 0.74401 | 6.64% |
| Verticle Double Sphere (Red)(8) | 0.882405 | 0.693005 | 0.03190 | 0.739486 | 6.71% |
| Verticle Double Sphere (Black)(8) | 0.882405 | 0.6927996 | 0.02530 | 0.739486 | 6.74% |

| Type | Distance | Terminal Velocity | Stand. Dev. from Terminal | Predicted* | Error |
|---|---|---|---|---|---|
| Verticle Double Sphere (Red)(9) | 1.07381 | 0.6862634 | 0.02646 | 0.721115 | 5.08% |
| Verticle Double Sphere (Black)(9) | 1.07381 | 0.6779655 | 0.01782 | 0.721115 | 6.36% |
| Verticle Double Sphere (Red)(10) | 1.55623 | 0.6434068 | 0.02345 | 0.688587 | 7.02% |
| Verticle Double Sphere (Black)(10) | 1.55623 | 0.6424083 | 0.01964 | 0.688587 | 7.19% |
| Verticle Double Sphere (Red)(11) | 1.99998 | 0.6156543 | 0.02247 | 0.668984 | 8.66% |
| Verticle Double Sphere (Black)(11) | 1.99998 | 0.611132 | 0.01884 | 0.668984 | 9.47% |
| Verticle Double Sphere (Red)(12) | 2.22849 | 0.6192461 | 0.02786 | 0.661232 | 6.78% |
| Verticle Double Sphere (Black)(12) | 2.22849 | 0.6160281 | 0.01593 | 0.661232 | 7.34% |
| Verticle Double Sphere (Red)(13) | 3.31512 | 0.5912805 | 0.03625 | 0.636585 | 7.66% |
| Verticle Double Sphere (Black)(13) | 3.31512 | 0.5907453 | 0.02128 | 0.636585 | 7.76% |
| Verticle Double Sphere (Red)(14) | 4.4378 | 0.5855548 | 0.02463 | 0.622195 | 6.26% |
| Verticle Double Sphere (Black)(14) | 4.4378 | 0.5785807 | 0.01803 | 0.622195 | 7.54% |

[p]

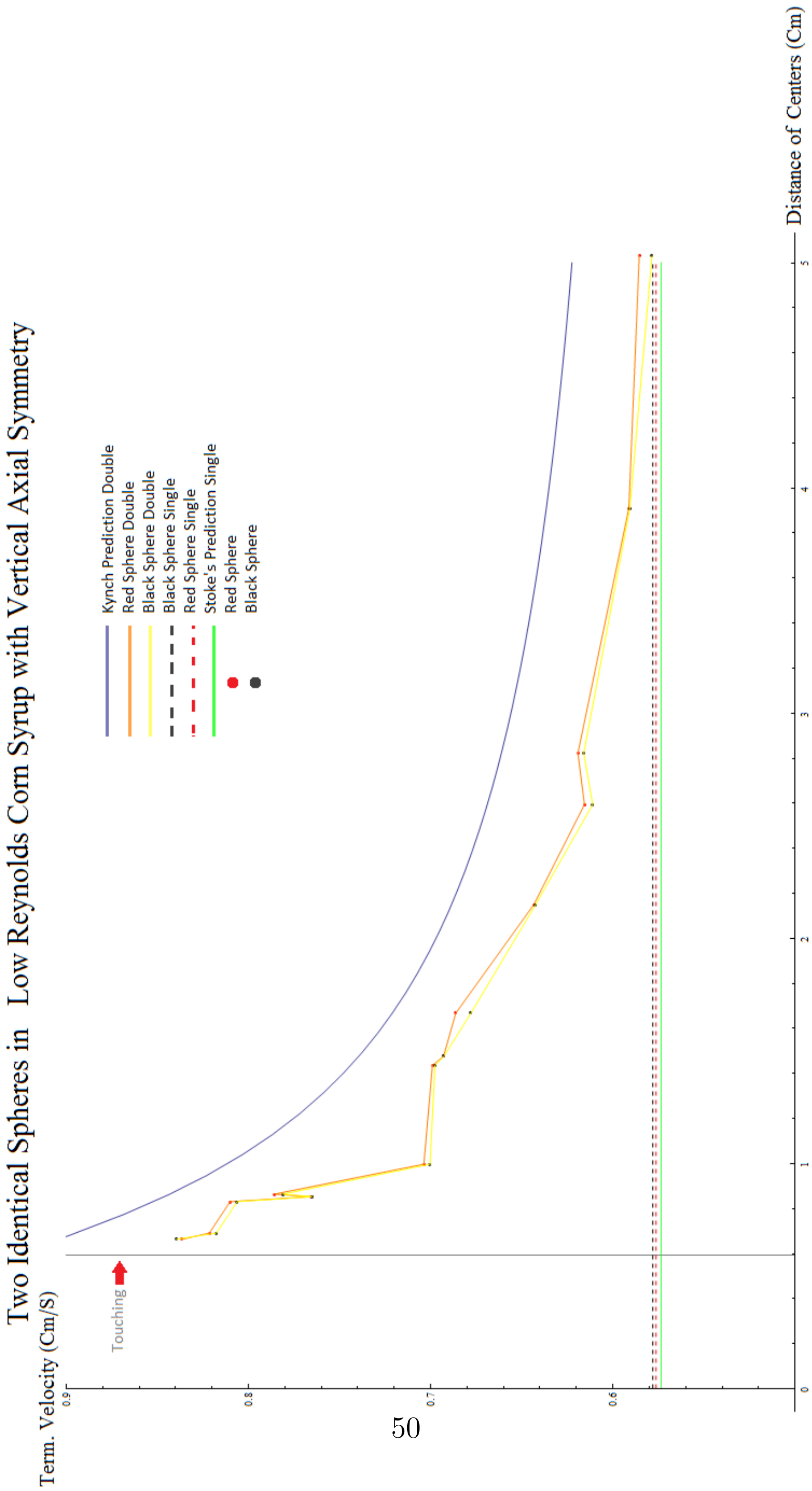





Figure 34: This data is better understood through visual representation

[p]

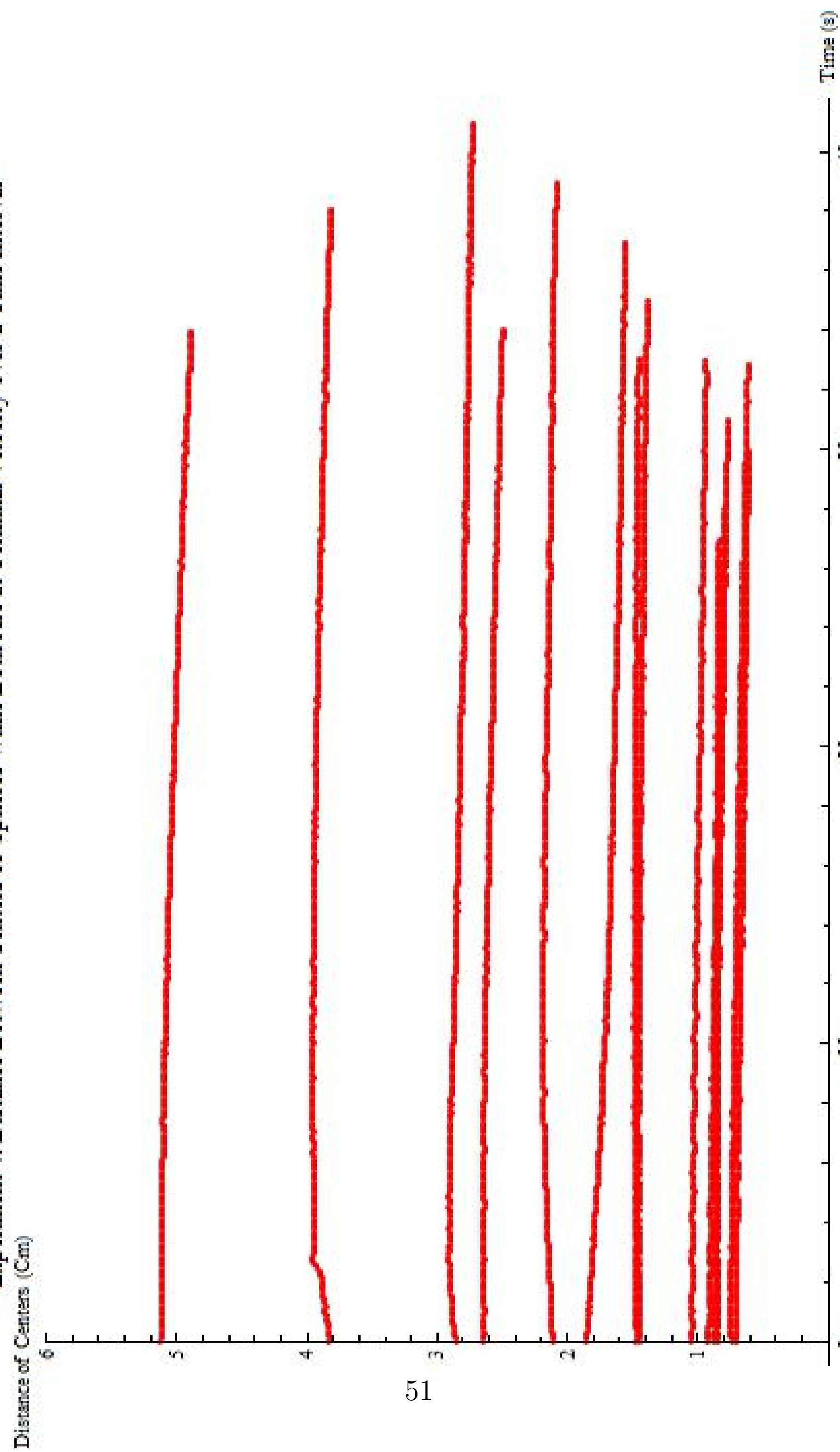



Figure 35: As predicted by Camassa, the distance between the spheres are relatively fixed with minor variance (This is corroborated by previous observations and theory)

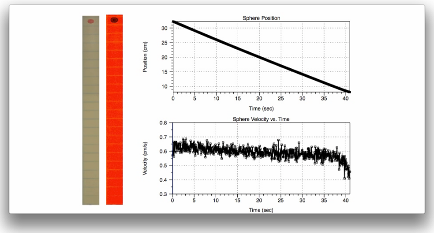


Figure 36: Single Sphere Red

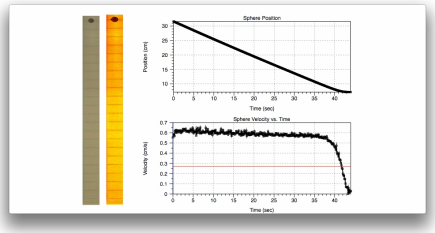


Figure 37: Single Sphere Black

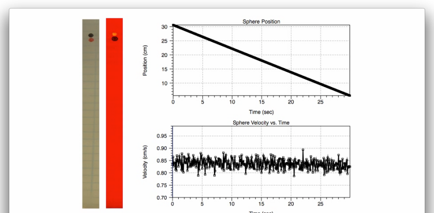


Figure 38: Double Sphere Red 0.069428 (cm)

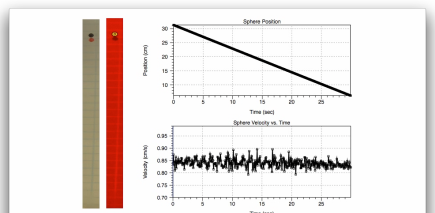


Figure 39: Double Sphere Black 0.069428 (cm)

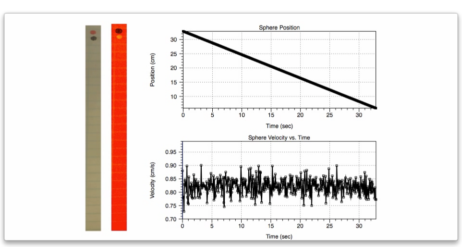


Figure 40: Double Sphere Red 0.0971035 (cm)

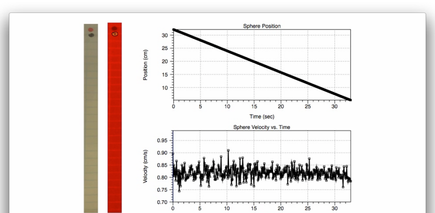


Figure 41: Double Sphere Black 0.0971035 (cm)

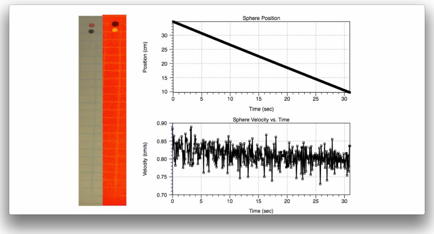



Figure 42: Double Sphere Red 0.236437 (cm)

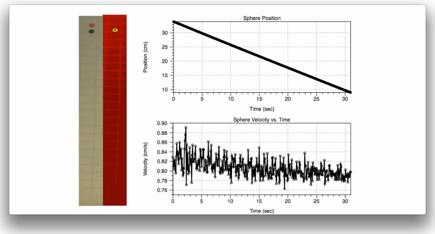



Figure 43: Double Sphere Black 0.236437 (cm)

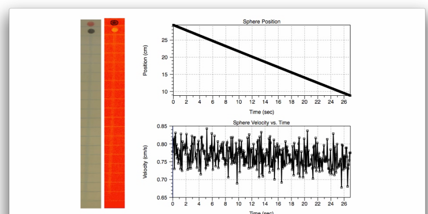



Figure 44: Double Sphere Red 0.258821 (cm)

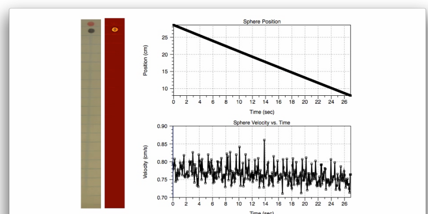



Figure 45: Double Sphere Black 0.258821 (cm)

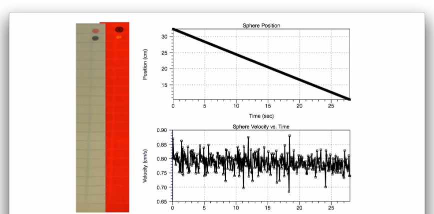



Figure 46: Double Sphere Red 0.269877 (cm)

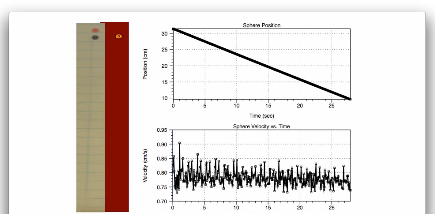



Figure 47: Double Sphere Black 0.269877(cm)

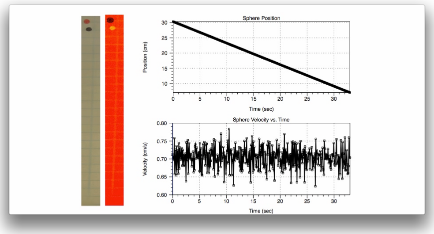


Figure 48: Double Sphere Red 0.40301 (cm)

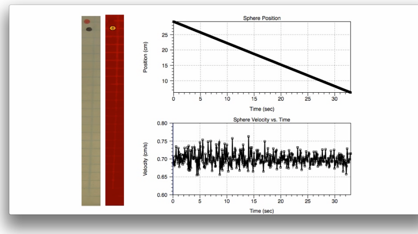


Figure 49: Double Sphere Black 0.40301 (cm)

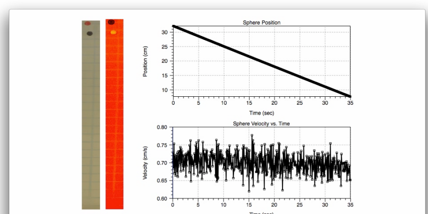


Figure 50: Double Sphere Red 0.841339(cm)

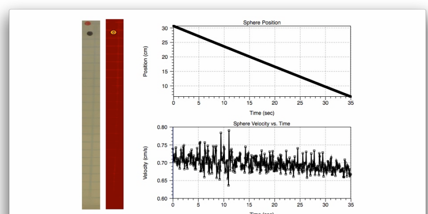


Figure 51: Double Sphere Black 0.841339 (cm)

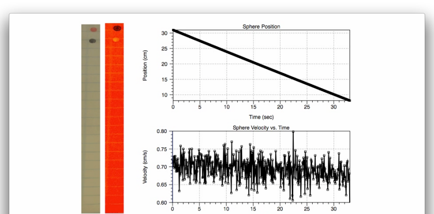


Figure 52: Double Sphere Red 0.882405 (cm)

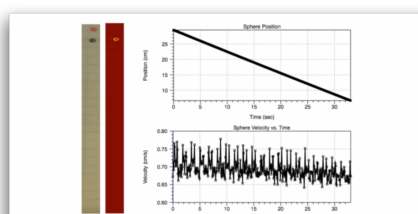


Figure 53: Double Sphere Black 0.882405(cm)

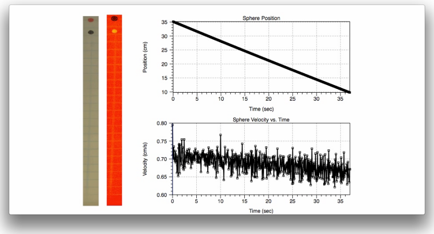


Figure 54: Double Sphere Red 1.07381(cm)

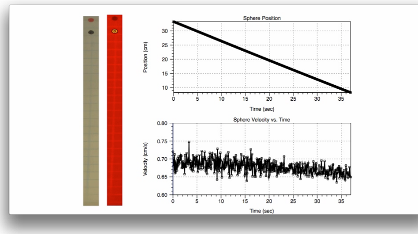


Figure 55: Double Sphere Black 1.07381 (cm)

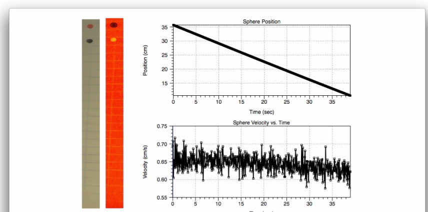


Figure 56: Double Sphere Red 1.55623(cm)

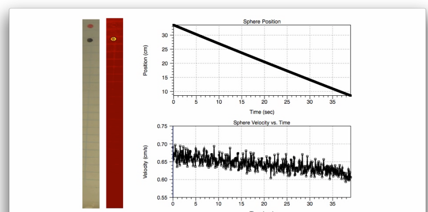


Figure 57: Double Sphere Black 1.55623 (cm)

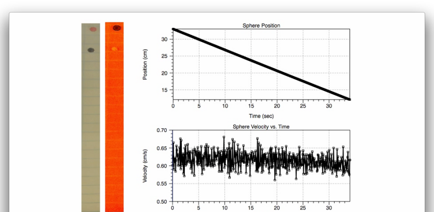


Figure 58: Double Sphere Red 1.99998 (cm)

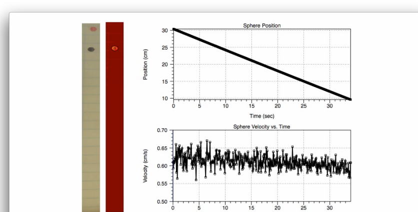


Figure 59: Double Sphere Black 1.99998 (cm)

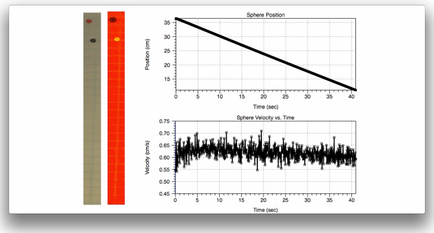


Figure 60: Double Sphere Red 2.22849 (cm)

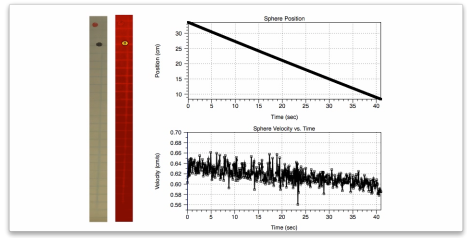


Figure 61: Double Sphere Black 2.22849 (cm)

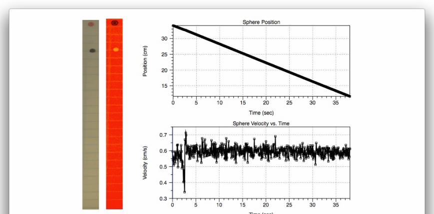


Figure 62: Double Sphere Red 3.31512 (cm)

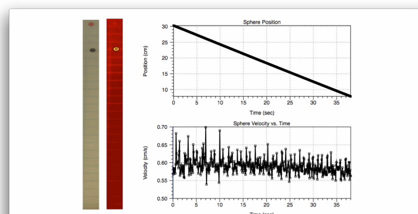


Figure 63: Double Sphere Black 3.31512(cm)

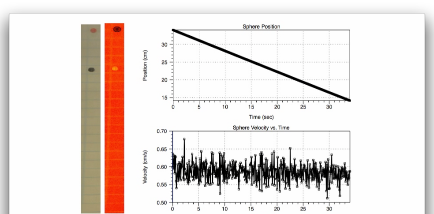


Figure 64: Double Sphere Red 4.4378 (cm)

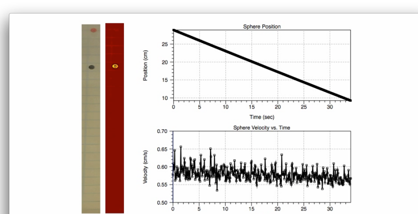


Figure 65: Double Sphere Black 4.4378 (cm)

**Issues:** The test went well and the only issues, as in previous experiments, was the necessity of better tracking and dropping. The tool that is used to drop the spheres does not compensate for the angle between the two spheres. It is still up to the experimenter to perfectly align them coaxially.

**Conclusion:** The experiment succeeded in providing a clear and definite result for coaxial spheres in low Reynolds, homogeneous fluid. The data looks great plotted, as when error is taken into account, seems completely within reason. It also show the issues with the Kynch prediction, seeming more like an upperbound for the experiment (although a vertical shift well aligns it with the points). With all this said, I think we have a strong enough of an experiment to move into our next regime: sharply stratified solutions.

# 18 Experiment Five: Two Identical Spheres in Stratified Corn Syrup (NaCl)

**Date:12/18/2015**

**Problem and Purpose:** Now we are moving into a stratified regime to act as a basis for our theory and to develop rudimentary observations. This experiment will also give us an understanding of what complications may arise from testing in a stratified regime and how a colored die effects our tracking.

**Materials:**
-D7000 Camera (Video Mode Recording)
-Small Plastic Cylinder Tank with Diameter of Approximately 10.75cm
-Top Corn Syrup of Density 1.36133 $g/cm^3$ and Viscosity 12.708$Pa.s$ Viscometer
-Bottom Corn Syrup of Density 1.38183 $g/cm^3$ and Viscosity 26.6427$Pa.s$ Viscometer
-Camera Mount
-Plastic Wrap
-Metal Mixing Pot
-Giant Plastic Spoon for stirring
-Designed Tool for Retrieval and Dropping

-Sphere Red, Radius(cm)=0.296545, Density($g/cm^3$) $\approx$ 2.26
-Sphere Black, Radius(cm)=0.296545, Density($g/cm^3$) $\approx$ 2.26

**Set-up:** Corn syrup from storage was poured into the metal mixing pot. Then a florescent yellow dye and salt was mixed into it. Then after 40 minutes of mixing, the density was taken. The corn syrup was then emptied into a 2000 ml container and covered in plastic wrap to prevent evaporation. The pot was then cleaned, and additional Corn syrup from storage was poured into it. Water was added to assist in making a high density separation and then the syrup was mixed for 40 minutes. The density was checked, the corn syrup was poured into another 2000 ml container, and then covered in a layer of plastic wrap to prevent evaporation. After a few days of degassing, the density and viscosity was retested on the two corn syrup samples. The spheres were recolored with marker. The camera was mounted, balanced and focused. A large beaker full of corn syrup from the tank that is used for: holding the retrieval tool, priming it for the drop, and holding the spheres, was put beside the tank. The metal mixing pot was placed beside the tank, full of water, for cleaning. Lastly, paper towels were procured from the bathrooms. The corn syrup mixed with table salt was poured into the tank first. Then, with the help of Claudia Falcon, the next layer of corn syrup, mixed with water, was poured into the tank, slowly, so as to create a sharp stratification. The ruler was stuck to the side of the tank via putty, the outside surface of the tank cleaned, and the liquid covered in plastic wrap.

**Procedure:** The experiment consisted of 7 drops, of which, 4 were considered suitable for analysis. An experiment was performed in the following manner. First, the spheres are submerged in the holding beaker. The focus and orientation of the camera is checked. Record button is pressed. The tool is submerged in holding beaker, and the spheres are sucked up. The tool is lifted from beaker with my hand cupping under the tool as to prevent corn syrup dripping on the table or the tank. The tool is submerged a few centimeters under the surface. Orientation of the tool in the liquid is checked. The spheres are slowly pushed out and distance adjusted. Slowly, the tool is removed and I back away as to prevent shadows. I wait for the spheres to stop falling, then stop the recording. Retrieve spheres using tool, being very careful to minimize damage to the interface. Drop spheres in a metal pot full of water. Clean the tool,and then the dry tool. Clean the spheres under water, then dry spheres. Put the spheres into a holding beaker. Repeat.

**Results:**

| Type | Single Red | Red-on-Black (1) Red-Black | Red-on-Black (2) Red-Black | Red-on-Black (3) Red-Black |
|---|---|---|---|---|
| Distance | NA | 0.3560335 | 0.4655336 | 2.486756 |
| Terminal Velocity Top | 1.20654 | 1.617611 — 1.59913 | 1.54164 — 1.528434 | 1.256683 — 1.230643 |
| Stand. Dev. of Terminal Velocity Top | 0.02035 | 0.06656 — 0.01364 | 0.09729 — 0.01956 | 0.09803 — 0.01533 |
| Predicted Top | 1.218262 | 1.752060 | 1.70338 | 1.39313 |
| Error Top | .97% | 8.31% — 9.56% | 10.49% — 11.45% | 10.86% — 13.20% |
| Terminal Velocity Bot | 0.56246 | 0.80601 — 0.81688 | 0.78888 — 0.80304 | 0.77535 — 0.75132 |
| Stand. Dev. of Terminal Velocity Bot | 0.00944 | 0.02346 — 0.01308 | 0.09700 — 0.01326 | 0.05993 — 0.01400 |
| Predicted Bot | 0.5439585 | 0.7823 | 0.760565 | 0.622036 |
| Error Bot | 3.29% | 2.94% — 4.23% | 3.59% — 5.29% | 19.77% — 17.21% |

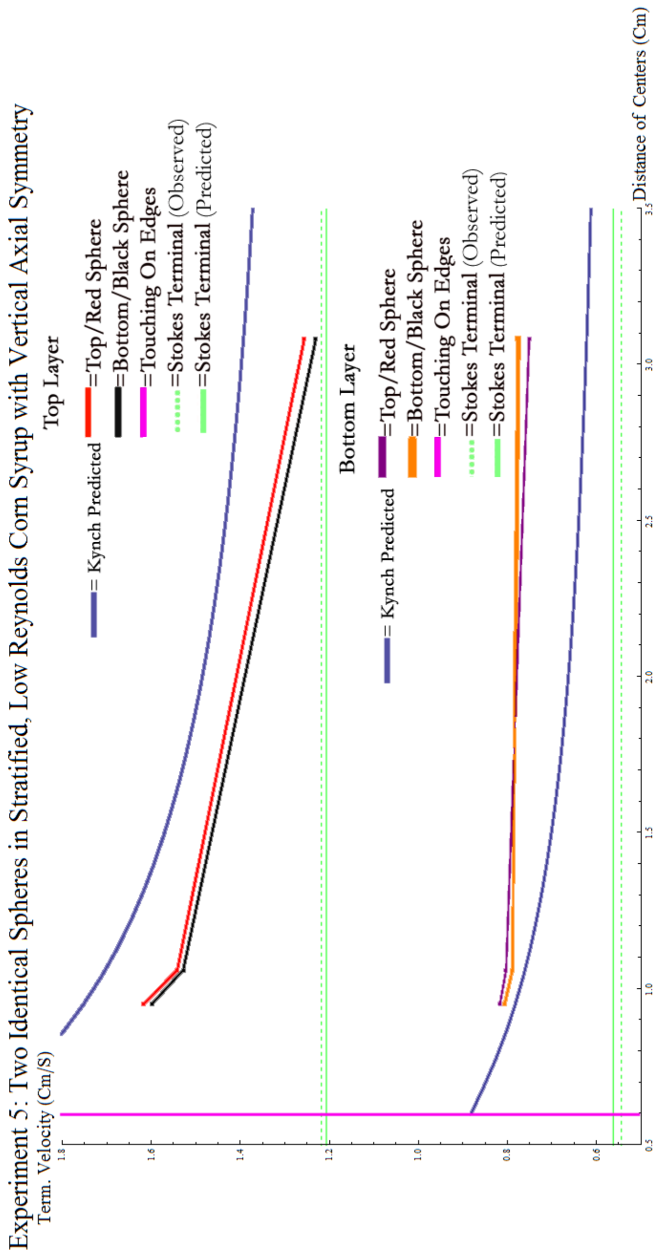



Figure 66: This data suggest that the forces making the sphere go faster is intensifying in the bottom layer

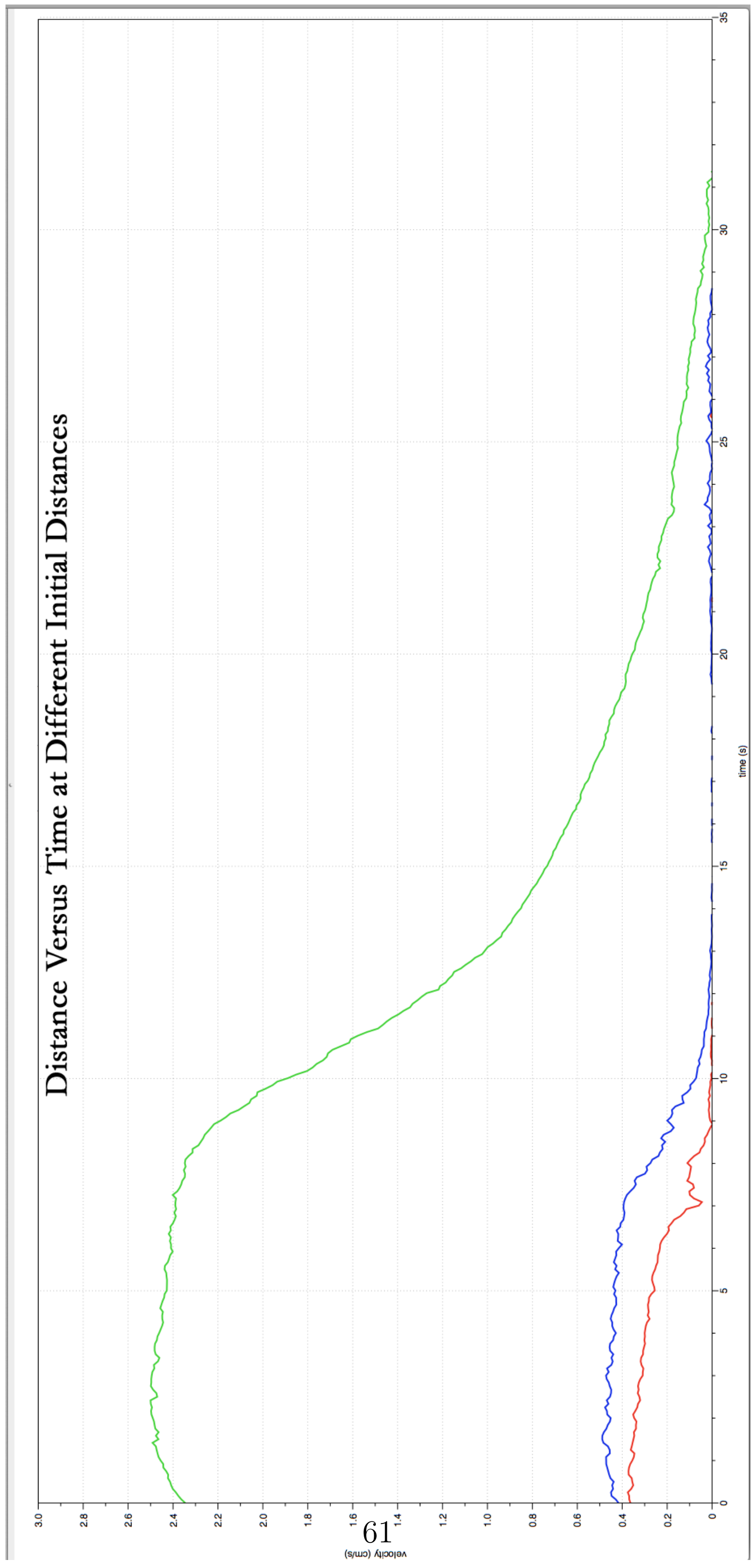





Figure 67: Wow! After puncturing the interface they approach each other!

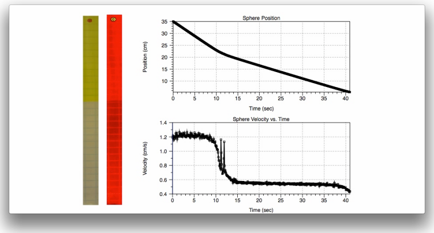



Figure 68: Single Sphere

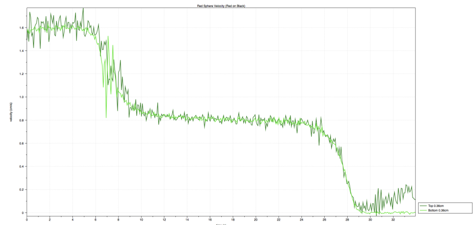


Figure 69: Double Sphere 0.3560335 (cm)

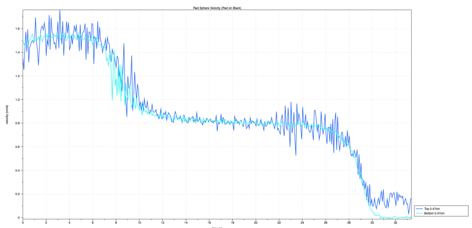


Figure 70: Double Sphere 0.4655336 (cm)

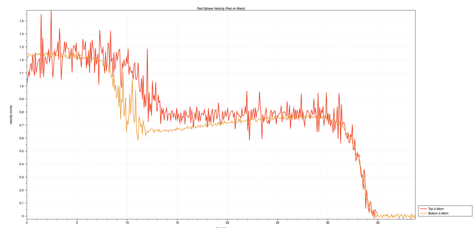


Figure 71: Double Sphere 2.486756 (cm)

**Issues:** The test should have been performed at an earlier hour in the day. Being around 1am, the experimenter was tired and only able to run the few test he did and was inhibited in performance. Likewise, the interface could have been more gently penetrated by the device, and the dying of the top layer did seem to help much and could have very well, had the opposite effect, making tracking more difficult. Overall a success, and when we run a future test to gather more data points, we can address these issues.

**Conclusion:** The experiment was a great success, especially considering the strong match between or stokes prediction and the actual values. Although, the Kynch prediction is still acting like an upper bound on the data, which seem to be off by a vertical shift. Also, the bottom layer having such a higher terminal then the Kynch prediction, which had previously been an upper bound, provides insight into what role entrainment is playing. Lastly, the amazing observation that spheres become closer in the bottom layer could be a focus for future experiments.

## 19 Experiment Six: Re-Test Two Identical Spheres in Stratified Corn Syrup (NaCl)

**Date:1/08/2016**

**Problem and Purpose:** This will be larger more expansive undertaking then experiment 5 (similar to how experiment 4 was to experiment 3), providing enough data points to establish a trend and behavior. We will also be using the data gathered from experiment 5 to improve testing and hopefully give us a good data set to begin developing theory.

**Materials:**
-D7000 Camera (Video Mode Recording)
-Small Plastic Cylinder Tank with Diameter of Approximately 10.75cm
-Top Corn Syrup of Density 1.38748 $g/cm^3$ and Viscosity 28.8448$Pa.s$ Viscometer @23.62 Celsius
-Bottom Corn Syrup of Density 1.37718 $g/cm^3$ and Viscosity 34.6511$Pa.s$ Viscometer @23.62 Celsius
-Camera Mount
-Plastic Wrap
-Metal Mixing Pot
-Giant Plastic Spoon for stirring
-Designed Tool for Retrieval and Dropping
-Sphere Red, Radius(cm)=0.296545, Density($g/cm^3$) $\approx$ 2.26
-Sphere Black, Radius(cm)=0.296545, Density($g/cm^3$) $\approx$ 2.26

**Set-up:** Corn syrup from storage was poured into the metal mixing pot. Then salt was mixed into it. Then, after 40 minutes of mixing, the density was taken. The corn syrup was then emptied into a 2000 ml container and covered in plastic wrap to prevent evaporation. The pot was then cleaned, and additional Corn syrup from storage was poured into it. Water was added to assist in making a high density separation and then the syrup was mixed for 40 minutes. The density was checked, and then it was then poured into another 2000 ml container, and covered in a layer of plastic wrap to prevent evaporation. After a few days of degassing, the density and viscosity was retested on the two corn syrup samples. The spheres were recolored with marker. The camera was mounted, balanced and focused. A large beaker full of corn syrup from the tank that is used for holding the retrieval tool, priming it for the drop, and holding the spheres was put beside the tank. The metal mixing pot was placed beside the tank, full of water, for cleaning. Lastly, paper towels were procured from the bathrooms. The corn syrup mixed with table salt was poured into the tank first. Then, with the help of Claudia Falcon, the next layer of corn syrup, mixed with water, was poured into the tank, slowly, so as to create a sharp stratification. The ruler was stuck to the side of the tank via putty, the outside surface of the tank cleaned, and the liquid covered in plastic wrap.

**Procedure:** The experiment consisted of 12 drops, of which 6 were considered suitable for analysis. An experiment was performed in the following manner. First, the spheres are submerged in the holding beaker. The focus and orientation of the camera is checked. Record button is pressed. The tool is submerged in holding beaker, and the spheres are sucked up. The tool is lifted from beaker with my hand cupping under the tool as to prevent corn syrup dripping on the table or the tank. The tool is submerged a few centimeters under the surface. Orientation of the tool in the liquid is checked. The spheres are slowly pushed out and distance adjusted. Slowly, the tool is removed and I back away as to prevent shadows. I wait for the spheres to stop falling, then stop the recording. Retrieve spheres using tool, being very careful to minimize damage to the interface. Drop spheres in a metal pot full of water. Clean the tool,and then the dry tool. Clean the spheres under water, then dry spheres. Put the spheres into a holding beaker. Repeat.

**Results/Issues/Conclusion:**

Unfortunately, the videos were all sideways and a new code to track the sphere was developed. However, after finding several shortcomings in the code that would require great difficulty in fixing (tracking distance and cropping issues), having the late realization that the densities were much to close for s strong stratification, and essentially needing another test due to only having 6 data points (two of which are single drops), I determined that it would be easier just to save the corn syrup and set-up another experiment later. The only conclusion that is of interest is that the sphere aproached each other (as in the previous experiment), but by approximately tenths of a centimeter. First, we should note that entrainment is most probably the cause (only happens after interaction with a sharp interface). Also, because of the much smaller approach of the two sphere to each other in this experiment, this indicates something related to the only two parameters that changed, the density and the viscosity. Lastly, I have a hunch the reduction in approach distance is strongly related to the viscosity since the difference in top and bottom viscosity is large in experiment 5 (difference of approx. 14 Pa.s) and small in experiment 6 (difference of approx. 6 Pa.s). Whereas the difference in density was approximately .01 in experiment 5 and .02 in experiment 6. Since the change in viscosity seems more dramatic, I'm inclined to see it as the main contributor.

# 20 Experiment Seven: New Salt, Two Identical Spheres in Stratified Corn Syrup (KI)

**Date:1/26/2016**

**Problem and Purpose:** As noted in experiment 6, this will be larger more expansive undertaking then experiment 5 (similar to how experiment 4 was to experiment 3), providing enough data points to establish a trend and behavior. We will also be using the data gathered from experiment 5 to improve testing and hopefully give us a good data set to begin developing theory. Also, with the advice of Professor Camassa, we will be using KI (Potassium Iodide) to create a larger density separation to see a more dramatic entrainment effect.

**Materials:**
-D7000 Camera (Video Mode Recording)
-Small Plastic Cylinder Tank with Diameter of Approximately 10.75cm
-Top Corn Syrup of Density 1.37781 $g/cm^3$ and Viscosity 30.774400$Pa.s$ Viscometer @23.05 Celsius
-Bottom Corn Syrup of Density 1.41578 $g/cm^3$ and Viscosity 26.494800$Pa.s$ Viscometer @23.05 Celsius
-Camera Mount
-Plastic Wrap
-Metal Mixing Pot
-Giant Plastic Spoon for stirring
-Designed Tool for Retrieval and Dropping
-Sphere Red, Radius(cm)=0.296545, Density($g/cm^3$) $\approx$ 2.26

-Sphere Black, Radius(cm)=0.296545, Density($g/cm^3$) $\approx 2.26$

**Set-up:** Corn syrup from storage was poured into the metal mixing pot. After 40 minutes of mixing, the density was taken and the corn syrup poured into a 2000 ml container and covered with plastic wrap. After cleaning the pot corn syrup from storage was added in along with approximately 55 ml of KI salt. After 40 minutes of stirring, taking a density and looking into the solubility of KI in water, I inferred that another 50 ml of KI would be safe to mix into solution without fear of over saturation. Then, after 40 minutes of mixing, the corn syrup was poured into a 2000 ml container and covered with plastic wrap. The solution was left for five days. Then on the day of the experiment, the viscosity and density of both corn syrups was gathered. The spheres were recolored with marker. The camera was mounted, balanced and focused. A large beaker full of corn syrup from the tank that is used for holding the retrieval tool, priming it for the drop, and holding the spheres was put beside the tank. The metal mixing pot was placed beside the tank, full of water, for cleaning. Lastly, paper towels were procured from the lab. The corn syrup mixed with KI was pored into the tank first. Then, with the help of Claudia Falcon, the next layer of corn syrup was poured into the tank, slowly, so as to create a stratification. The ruler was stuck to the side of the tank via putty, the outside surface of the tank cleaned, and the liquid covered in plastic wrap.

**Procedure:** The experiment consisted of 10 drops of the 2.6 density spheres, of which, 9 were considered suitable for analysis. An experiment was performed in the following manner. First, the spheres are submerged in the holding beaker. The focus and orientation of the camera is checked. Record button is pressed. The tool is submerged in holding beaker, and the spheres are sucked up. The tool is lifted from beaker with my hand cupping under the tool as to prevent corn syrup dripping on the table or the tank. The tool is submerged a few centimeters under the surface. Orientation of the tool in the liquid is checked. The spheres are slowly pushed out and distance adjusted. Slowly, the tool is removed and I back away as to prevent shadows. I wait for the spheres to stop falling, then stop the recording. Retrieve spheres using tool, being very careful to minimize damage to the interface. Drop spheres in a metal pot full of water. Clean the tool,and then dry the tool. Clean the spheres under water, then dry spheres. Put the spheres into a holding beaker. Repeat.

**Results*:**

| Type | Distance | Terminal Velocity) | Stand. Dev. from Terminal | Predicted |
|---|---|---|---|---|
| Single Red 1 | NA | 0.5129926 — 0.5554601 | 0.009147 — 0.008163 | 0.485941 — 0.540139 |
| Single Black 1 | NA | 0.5091437 — 0.5535152 | 0.02646 — 0.013252 | 0.485941 — 0.540139 |
| Double Red-on-Black 1 Red | 1.84107 | 0.5666663 — 0.5739903 | 0.011689 — 0.01015 | 0.680448 — 0.756339 |
| Double Red-on-Black 1 Black | 1.84107 | 0.5690487 — 0.5792552 | 0.011062 — 0.010401 | 0.680448 — 0.756339 |
| Double Red-on-Black 2 Red | 1.55457 | 0.5934052 — 0.5607497 | 0.025621 — 0.024766 | 0.703414 — 0.781866 |
| Double Red-on-Black 2 Black | 1.55457 | 0.5959242 — 0.5878676 | 0.014607 — 0.014690 | 0.703414 — 0.781866 |
| Double Red-on-Black 3 Red | 0.92246 | 0.6733766 — 0.6197672 | 0.024362 — 0.011448 | 0.79678 — 0.885647 |
| Double Red-on-Black 3 Black | 0.92246 | 0.6731844 — 0.6279668 | 0.019373 — 0.027474 | 0.79678 — 0.885647 |
| Double Red-on-Black 4 Red | 0.57177 | 0.7407136 — 0.7897054 | 0.018860 —0.018160 | 0.900967 — 1.00145 |
| Double Red-on-Black 4 Black | 0.57177 | 0.7410768 — 0.7967816 | 0.022053 — 0.019594 | 0.900967 — 1.00145 |

| | | | | |
|---|---|---|---|---|
| Double Red-on-Black 5 Red | 2.28915 | 0.5513091 — 0.5690348 | 0.012556 — 0.01095917 | 0.655545 — 0.728659 |
| Double Red-on-Black 5 Black | 2.28915 | 0.5553703 — 0.5778868 | 0.011431 —0.010663 | 0.655545 — 0.728659 |
| Double Red-on-Black 6 Red | 2.04872 | 0.5636054 — 0.5791662 | 0.016237 —0.012141 | 0.667612 — 0.742072 |
| Double Red-on-Black 6 Black | 2.04872 | 0.5661987 — 0.5792844 | 0.012655 —0.015428 | 0.667612 — 0.742072 |

*For columns with (-) between numbers, indicates Top(-)Bottom

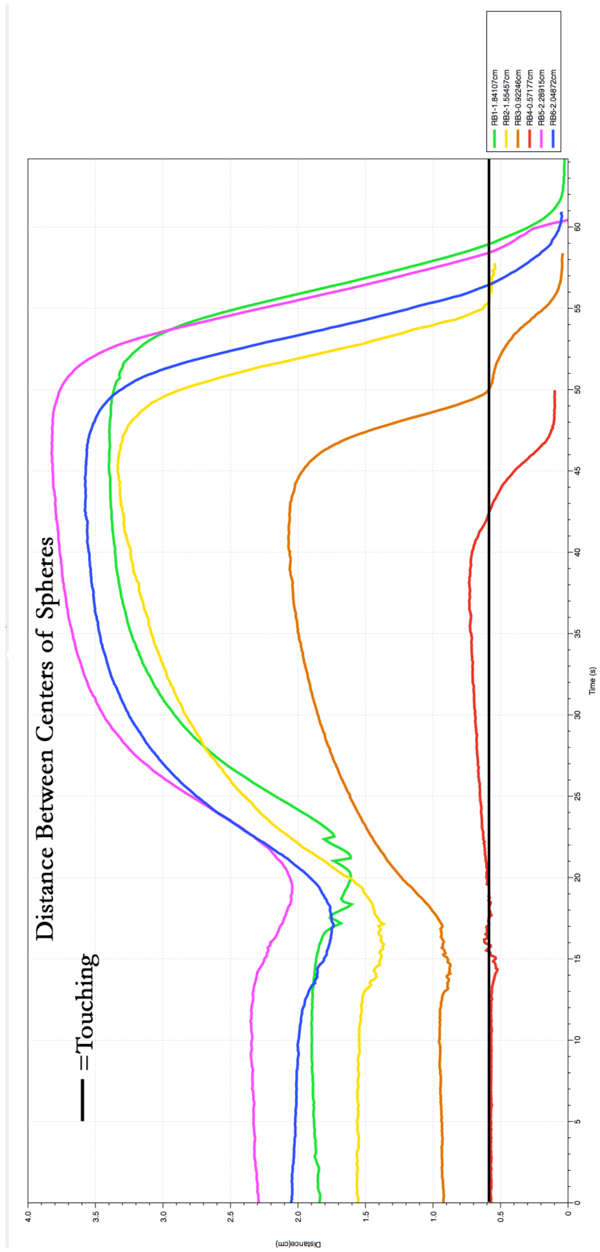



Figure 72: Wow! Now the spheres are separating from each other!

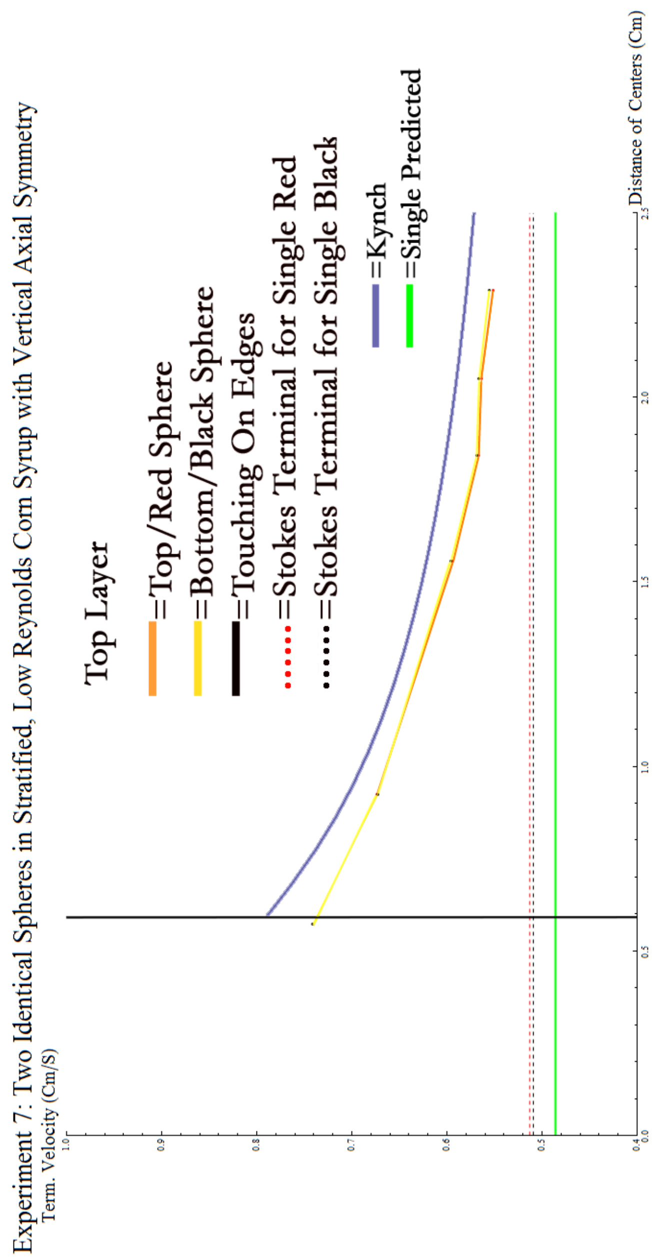



Figure 73: Nothing extraordinary here compared to (NaCl)

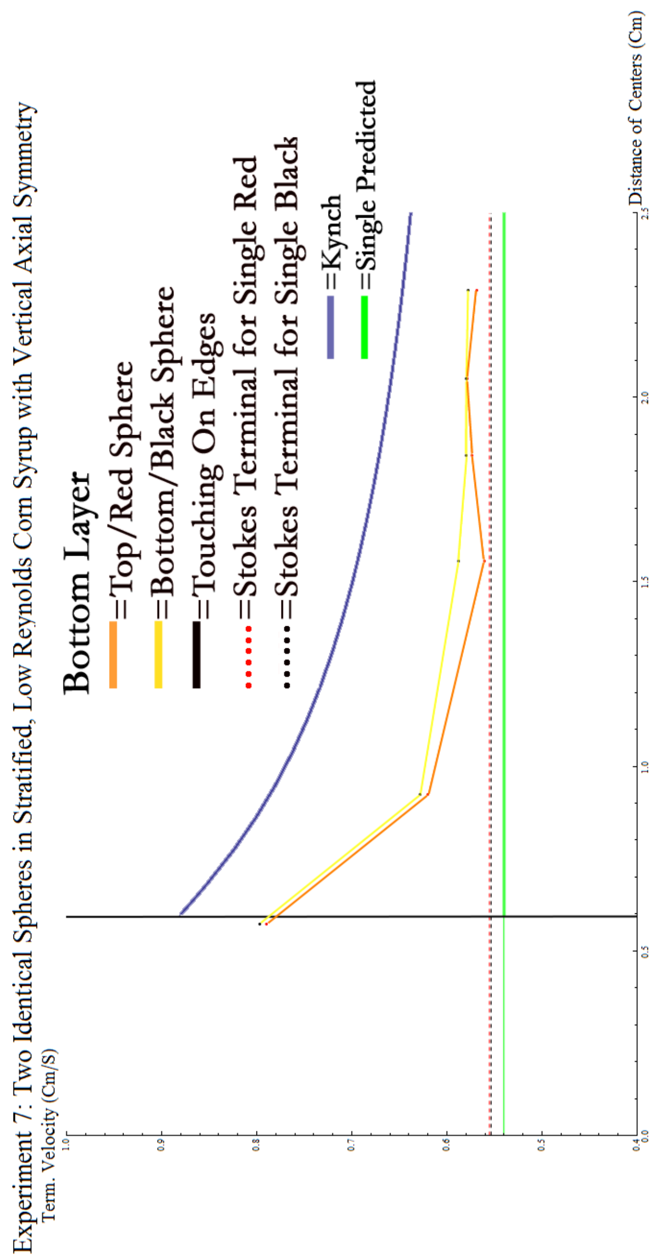



Figure 74: We are now seeing the spheres travel at different Term. Velocities

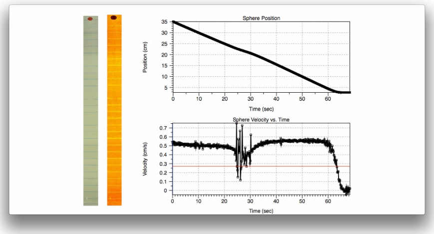



Figure 75: Single Sphere

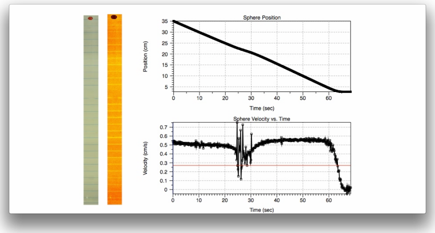



Figure 76: Double Sphere 0.3560335 (cm)

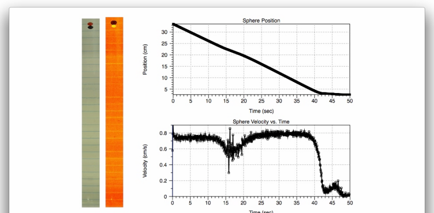



Figure 77: Double Sphere 0.4655336 (cm)

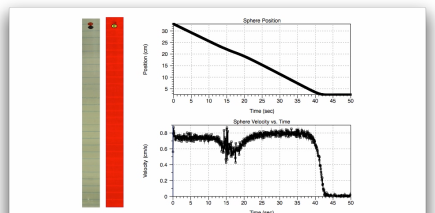



Figure 78: Double Sphere 2.486756 (cm)

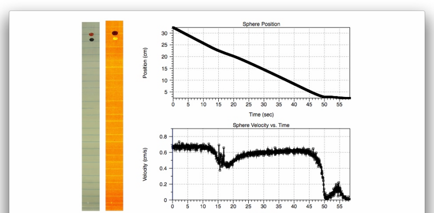



Figure 79: Double Sphere 0.4655336 (cm)

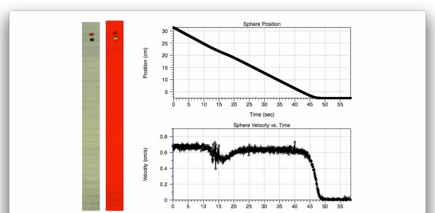



Figure 80: Double Sphere 2.486756 (cm)

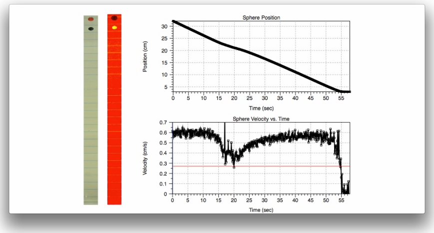


Figure 81: Double Sphere 0.4655336 (cm)

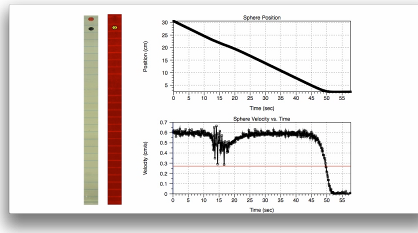


Figure 82: Double Sphere 2.486756 (cm)

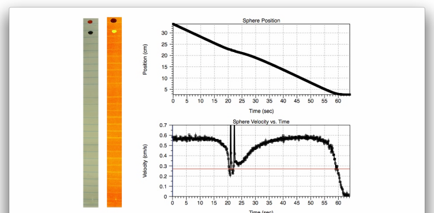


Figure 83: Double Sphere 0.4655336 (cm)

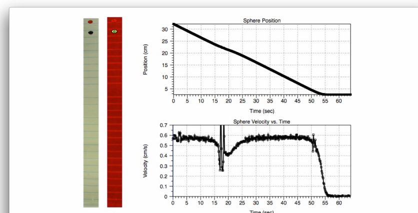


Figure 84: Double Sphere 2.486756 (cm)

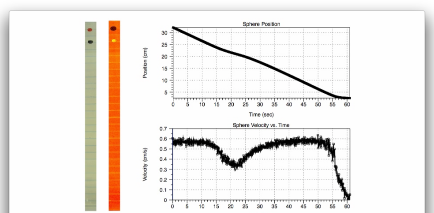


Figure 85: Double Sphere 0.4655336 (cm)

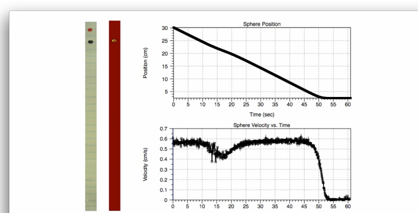


Figure 86: Double Sphere 2.486756 (cm)

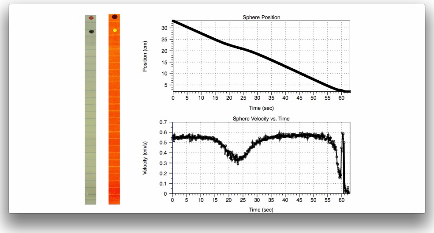



Figure 87: Double Sphere 0.4655336 (cm)

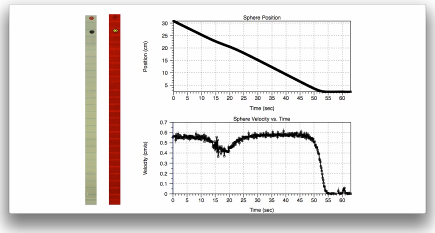



Figure 88: Double Sphere 2.486756 (cm)

**Issues:** The test was a success and the only issues, as in previous experiments, is the constant need for better tracking and dropping. Also, the deviation from predicted terminal was higher then expected, but may be due to temperature variations. It would be nice to pin down why exactly the predicted is slower then what we observe in almost every experiment. Lastly, at the advice of Dr. Camassa, to create a simpler scenario to observe behavior and to have data tat can be run in Claudia's simulation, I will need to get nearly identical viscosity.

**Conclusion:** The experiment providing some amazing conclusions. First, and foremost, the spheres are now separating from each other. Is this a consequence of the different salt? Second, we are noticing a very consistent behavior/shape on the distance plots. Can we predict that? The third and least understood observation is that within a certain range, the top sphere is going slower then the bottom sphere. Why are they flip flopping?. This needs to be looked into further.

## 21 Experiment Eight: Test Two Identical Spheres in Stratified Corn Syrup (KI) with Similar Viscosity

**Date:2/10/2016**

**Problem and Purpose:** Experiment 7 was a strong experiment but, at the advice of Dr. Camassa, similar viscosity will eliminate some variables, making understanding the basic behavior easier. Likewise, Claudia Falcon's simulation code for single particle through a stratified fluid require identical viscosity. So by satisfying this, we will be able to simulate and test our experiments against those simulations. As before, more data is always a good thing, and since we are adding water to the top layer to create similar viscosity, the density separation will be greater, allowing for more dramatic stratification.

**Materials:**
-D7000 Camera (Video Mode Recording)
-Small Plastic Cylinder Tank with Diameter of Approximately 10.75cm
-Top Corn Syrup of Density 1.42608 $g/cm^3$ and Viscosity 26.7944$Pa.s$ Viscometer @23.05 Celsius
-Bottom Corn Syrup of Density 1.37560 $g/cm^3$ and Viscosity 2641.94$Pa.s$ Viscometer @23.05 Celsius
-Camera Mount
-Plastic Wrap
-Metal Mixing Pot
-Giant Plastic Spoon for stirring
-Designed Tool for Retrieval and Dropping
-Sphere Red, Radius(cm)=0.296545, Density($g/cm^3$) $\approx$ 2.26
-Sphere Black, Radius(cm)=0.296545, Density($g/cm^3$) $\approx$ 2.26

**Set-up:** After borrowing a mortar and pestle from the neighboring lab, I used 45 minutes to grind 160 ml of granulated KI. Corn syrup from storage was poured into the metal mixing pot. After 40 minutes of mixing, the density was taken and the corn syrup poured into a 2000 ml container and covered with plastic wrap. After cleaning the pot corn syrup from storage was added in along with approximately 120 of KI salt. After 40 minutes of stirring, I pored the plain corn syrup into a 2000 ml container and covered with plastic wrap. Then I took the density and viscosity of both. Then the corn syrup mixed with KI was poured into the experimental tank and covered with plastic wrap. The solution was left for four days. Then on the day of the experiment, the viscosity and density of both corn syrups was taken again. ( the viscosity matching is detailed below). The spheres were recolored with marker. The camera was mounted, balanced and focused. I poured top layer into a large beaker for holding the retrieval tool, priming it for the drop, and holding the spheres. The metal mixing pot was placed beside the tank, full of water, for cleaning. Lastly, paper towels were procured from the lab. Then the next layer of corn syrup mixed with water was poured from the 2000ml container into the tank, slowly, so as to create a stratification. The ruler was stuck to the side of the tank via putty, the outside surface of the tank cleaned, and the liquid covered in plastic wrap.

| Layer | Temperature (C) | Density (g/cm^3) | Viscosity (m.Pa.s) |
|---|---|---|---|
| **Bot 2/10/16** | 23.05 | 1.42608 | 2641.94 |
| | 20.00 | 1.42806 | 3466.0901 |
| | 25.00 | 1.42479 | 2115.0197 |
| **Top 2/6/16** | 22.39 | 1.37785 | NA |
| | 20.00 | 1.37923 | 4121.2089 |
| | 25.00 | 1.37613 | 2498.8523 |
| **Top* 2/10/16** | 23.05 | 1.37740 | NA |
| | 20.00 | 1.37927 | NA |
| | 25.00 | 1.37618 | 2505.4207 |
| **Top** 2/10/16 +5 ml of Water** | 23.05 | 1.37720 | NA |
| | 20.00 | 1.37905 | 4102.5269 |
| | 25.00 | 1.37438 | NA |
| **Top*** 2/10/16 +12 ml of Water** | 23.05 | 1.37560 | 2679.43532 |
| | 20.00 | 1.37749 | 3519.9526 |
| | 25.00 | 1.37438 | 2146.5626 |

Figure 89: This data is better understood through visual representation

**Procedure:** The experiment consisted of 11 drops of the 2.6 density spheres, of which, none were considered suitable for analysis. An experiment was performed in the following manner. First, the spheres are submerged in the holding beaker. The focus and orientation of the camera is checked. Record button is pressed. The tool is submerged in holding beaker, and the spheres are sucked up. The tool is lifted from beaker with my hand cupping under the tool as to prevent corn syrup dripping on the table or the tank. The tool is submerged a few centimeters under the surface. Orientation of the tool in the liquid is checked. The spheres are slowly pushed out and distance adjusted. Slowly, the tool is removed and I back away as to prevent shadows. I wait for the spheres to stop falling, then stop the recording. Retrieve spheres using tool, being very careful to minimize damage to the interface. Drop spheres in a metal pot full of water. Clean the tool,and then dry the tool. Clean the spheres under water, then dry spheres. Put the spheres into a holding beaker. Repeat.

**Results/Issues/Conclusion:**

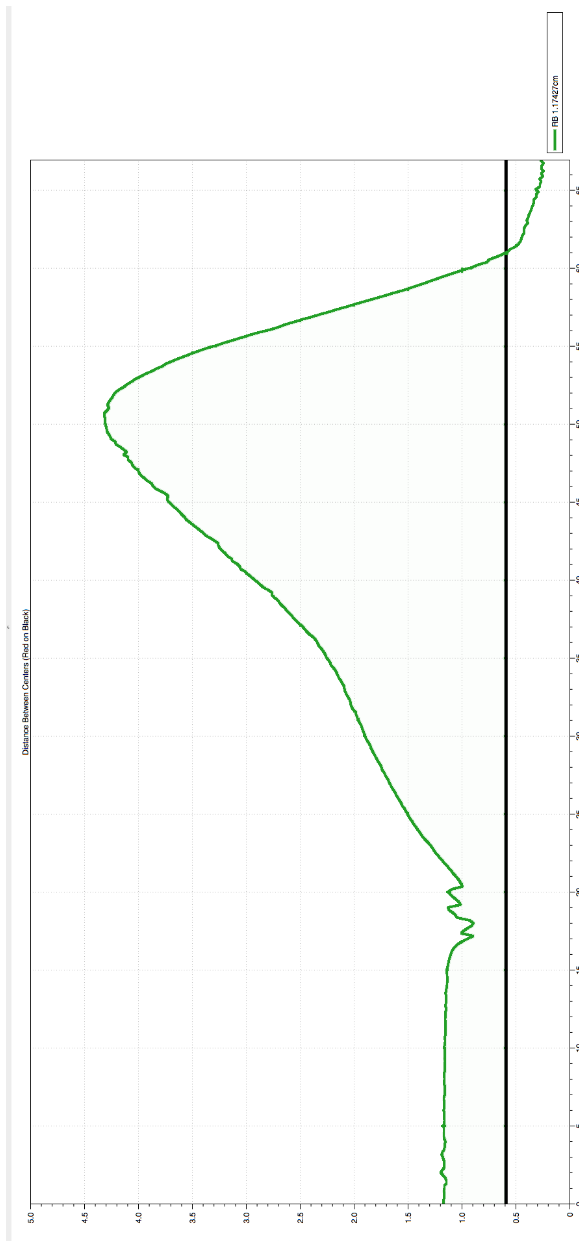



Figure 90: This data is better understood through visual representation

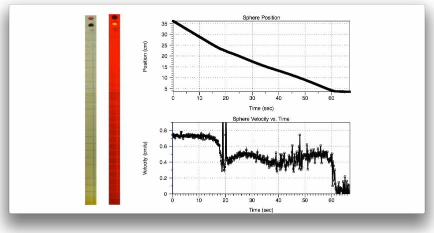


Figure 91: Double Sphere 0.4655336 (cm)

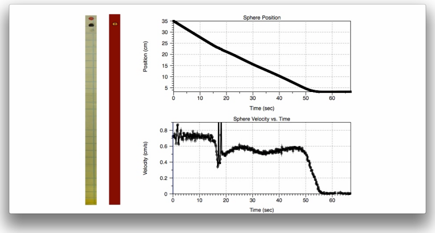


Figure 92: Double Sphere 2.486756 (cm)

Unfortunately, after attempting to analyze the videos, it became obvious that there was a linear stratification in the bottom layer. I theorize that the undissolved salt (since it was grounded), had lied in the bottom of the tank and then, between the initial mixing and test day, slowly dissolved into the bottom. This then created a lower higher density in the bottom of the bottom layer.

## 22 Experiment Nine: Re-Test Two Identical Spheres in Stratified Corn Syrup (KI)

**Date:2/27/2016**

**Problem and Purpose:** Experiment 8 was a failure that still resulted in some interesting observations. Regardless, we still need similar viscosity. As noted before, this will eliminate some variables, making understanding the basic behavior easier and satisfies the parameters for Claudia's simulation. Also as noted, more data is always a good thing, and since we are adding water to the top layer to create the similar viscosity, the density separation will be greater, allowing for more dramatic stratification. Lastly, and most importantly, this will provide the parameters necessary to apply the observations made on the velocity sphere from a UNC student

**Materials:**
-D7000 Camera (Video Mode Recording)
-Small Plastic Cylinder Tank with Diameter of Approximately 10.75cm
-Top Corn Syrup of Density 1.42450 $g/cm^3$ and Viscosity 27.2640$Pa.s$ Vis-

cometer @22.48* Celsius
-Bottom Corn Syrup of Density 1.37419 $g/cm^3$ and Viscosity 26.9517$Pa.s$ Viscometer @22.48* Celsius
-Camera Mount
-Plastic Wrap
-Metal Mixing Pot
-Giant Plastic Spoon for stirring
-Designed Tool for Retrieval and Dropping
-Sphere Red, Radius(cm)=0.296545, Density($g/cm^3$) $\approx$ 2.26
-Sphere Black, Radius(cm)=0.296545, Density($g/cm^3$) $\approx$ 2.26

**Set-up:** After borrowing a mortar and pestle from the neighboring lab, is spent 45 minutes grinding 160 ml of granualted KI. corn syrup from storage was poured into the metal mixing pot. After 40 minutes of mixing, the density was taken and the corn syrup poured into a 800ml container and covered with plastic wrap. After cleaning the pot corn syrup from storage was added in along with approximately 120 of KI salt. After 40 minutes of stirring, I took the density and viscosity of both. Then the corn syrup mixed with KI was poured into the experimental tank and covered with plastic wrap. The solution was left for four days. Then on the day of the experiment, the viscosity and density of both corn syrups was taken again. The spheres were recolored with marker. The camera was mounted, balanced and focused. A large beaker full of corn syrup from the tank that is used for holding the retrieval tool, priming it for the drop, and holding the spheres was put beside the tank. The metal mixing pot was placed beside the tank, full of water, for cleaning. Lastly, paper towels were procured from the lab. Then the next layer of corn syrup mixed with water was poured from the 800ml container into the tank, slowly, so as to create a stratification. The ruler was stuck to the side of the tank via putty, the outside surface of the tank cleaned, and the liquid covered in plastic wrap.

***Important Note: Post experiment** In previous experiments with the stokes term velocity prediction, we predicted terminal velocity within 1% of the experimental observation. This gave confidence to both the experimental technique and the lab tools. So whenever I calculated the predicted terminal velocity $\approx 0.56cm/s$ as compared to the experimental of $\approx 0.62cm/s$ (a error of 11%). I realized something must be off. I have a strong suspicion it is the temperature. The only reason I assume this is: first by noticing that, unlike in past experiments, I had taken the temperature 12 hours before running the experiment, second that the temperature would only need to change from 22.48 C to 23.60 C. This is probable since experiments have ranged from NA to NA. Regardless, we need to be certain of these factors to apply our mathematical models. So until I can run an experiment in a temperature bath, we will treat it as if the temperature was 23.6 C.

| Layer | Temperature (C) | Density (g/cm^3) | Viscosity (m.Pa.s) |
|---|---|---|---|
| Bot | 22.48 | 1.42450 | 2726.40 |
| | 20.00 | 1.42606 | 3465.4316 |
| | 25.00 | 1.42273 | 1975.4509 |
| Top | 25.00 | 1.37332 | 2403.7684 |
| Top* +15 ml of Water | 25.00 | 1.37479 | 2264.1231 |
| Top** +10 ml of Water | 25.00 | 1.37392 | 2162.4143 |
| Top*** +12 ml of Water | 22.48 | 1.37419 | 2695.17 |
| | 20.00 | 1.37583 | 3392.4436 |
| | 25.00 | 1.37259 | 2008.9637 |

Figure 93: Process of creating similar viscosity

**Procedure:** The experiment consisted of 17 drops of the 2.6 density spheres, of which, all were considered suitable for analysis. An experiment was performed in the following manner, spheres are submerged in holding beaker. Focus and orientation of camera is checked. Record button is pressed. Tool is submerged in holding beaker, and spheres are sucked up. Tool is lifted from beaker and hand cups bottom to prevent dripping on table and tank. Tool is submerged. Orientation of tool in liquid is checked. Spheres are slowly pushed out and distance adjusted. Slowly tool is removed and I back away as to prevent shadows. I wait for spheres to stop falling, then Stop the recording. Retrieve spheres using tool, being very careful to minimize damage to the interface. Drop spheres in metal pot full of water. Clean tool, then dry tool. Clean spheres under water, then dry spheres. Put spheres into holding beaker. Repeat.

**Results**

| Type | Color | Initial Distance | Term Velocity Top | Term Velocity Bot |
|---|---|---|---|---|
| R1 | Red | NA | 0.62677 | 0.542423 |
| B1 | Black | NA | 0.62647 | 0.543191 |
| B2 | Black | NA | 0.629834 | 0.541156 |
| BR1 | Red | 0.0753901 | 0.944265 | 0.799721 |
| BR1 | Black | 0.0753901 | 0.936975 | 0.7976 |
| RB1 | Red | 0.0787214 | 0.921535 | 0.780833 |
| RB1 | Black | 0.0787214 | 0.911521 | 0.780117 |
| RB2 | Red | 0.10028 | 0.888949 | 0.727493 |
| RB2 | Black | 0.10028 | 0.889808 | 0.732412 |
| RB3 | Red | 0.62501 | 0.74834 | 0.629409 |
| RB3 | Black | 0.62501 | 0.751506 | 0.622925 |
| RB4 | Red | 0.760637 | 0.755427 | 0.592296 |
| RB4 | Black | 0.760637 | 0.755477 | 0.602361 |
| RB5 | Red | 0.900982 | 0.735297 | 0.571736 |
| RB5 | Black | 0.900982 | 0.738568 | 0.589432 |
| RB6 | Red | 0.988434 | 0.739066 | 0.58471 |
| RB6 | Black | 0.988434 | 0.735851 | 0.596217 |
| RB7 | Red | 4.00516 | 0.618821 | 0.546104 |
| RB7 | Black | 4.00516 | 0.617549 | 0.563281 |

Figure 94: Predicted values are excluded due to inaccurate temperature

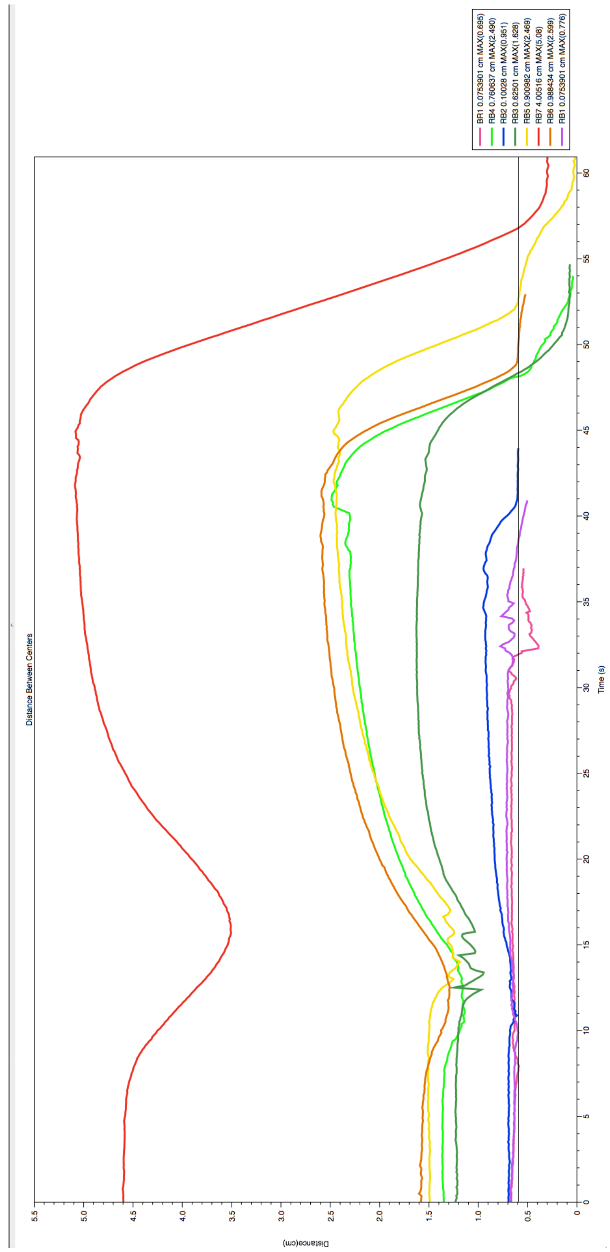



Figure 95: As in Experiment 7, we are seeing regular behavior in these Distance Plots

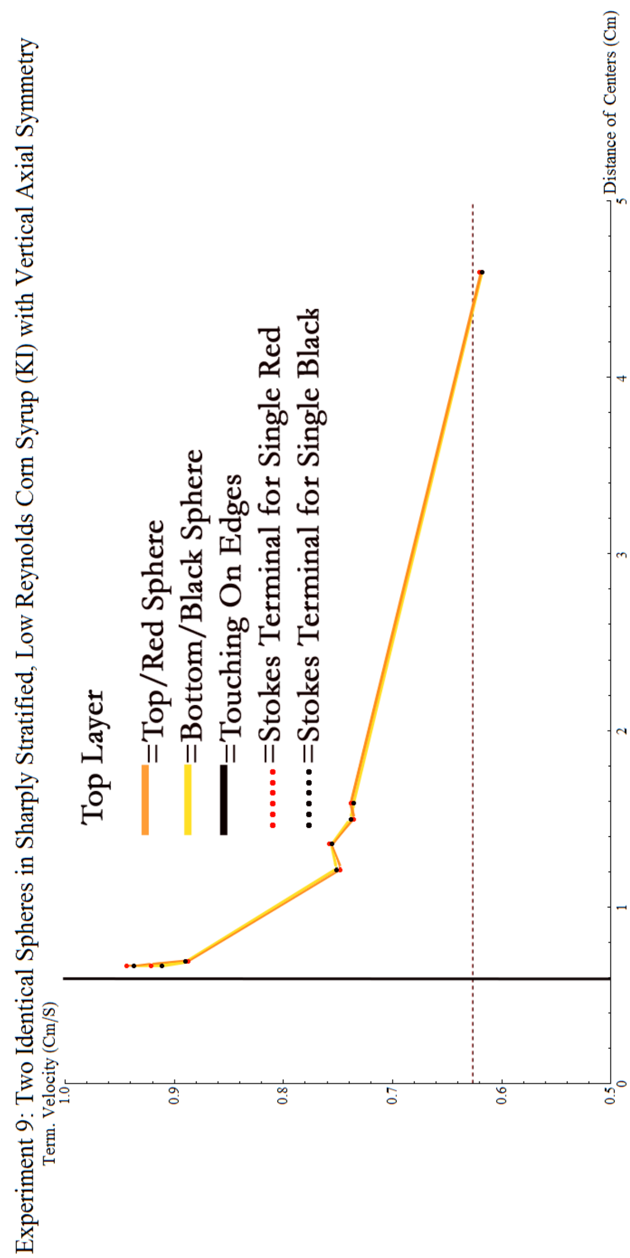



Figure 96: Plot of Data in Top Layer

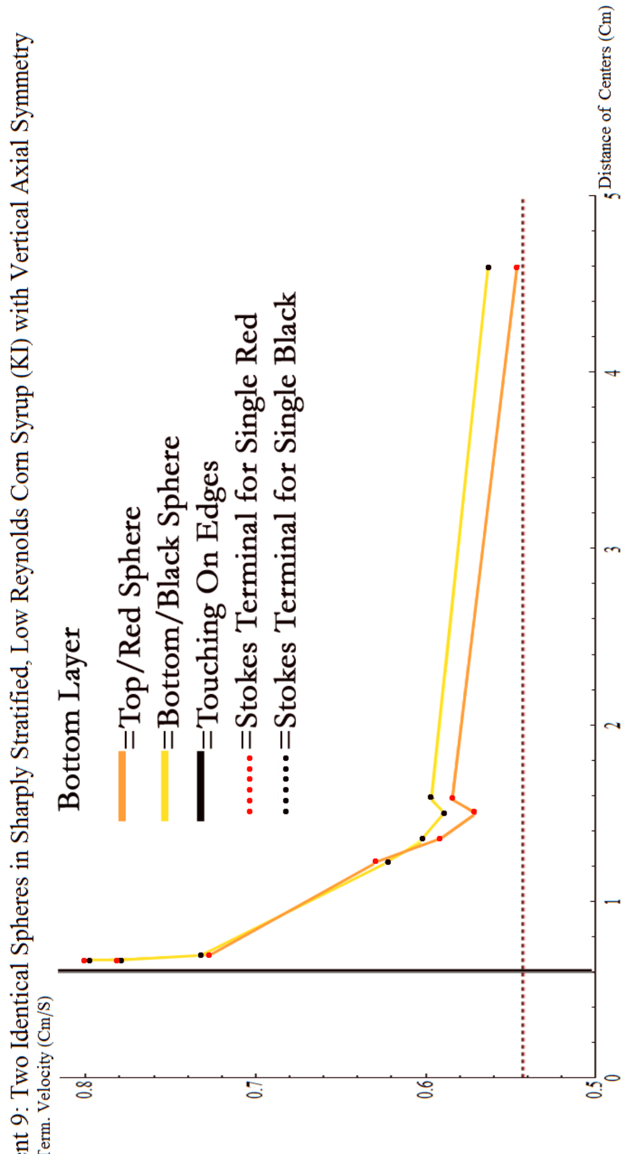



Figure 97: Plot of Data in Bottom Layer

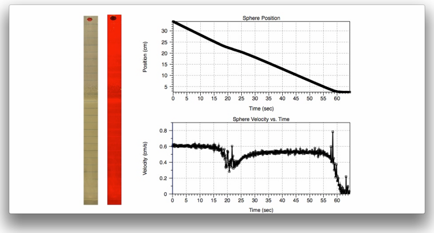


Figure 98: R1

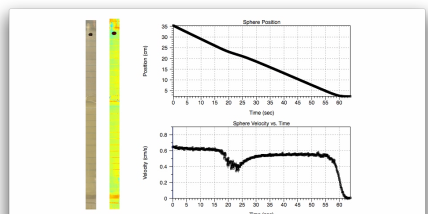


Figure 99: B1)

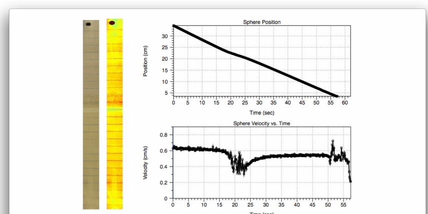


Figure 100: B2

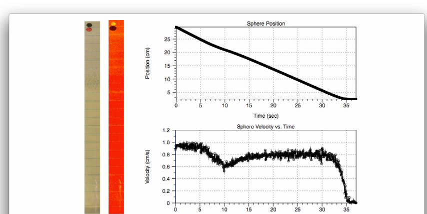


Figure 101: BR1R

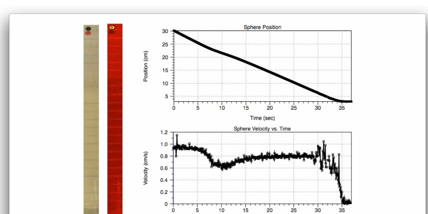


Figure 102: BR1B)

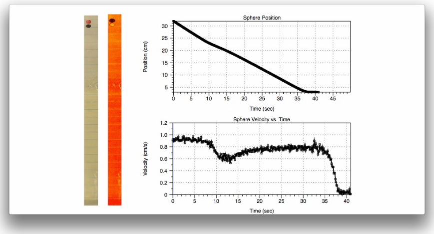


Figure 103: RB1R

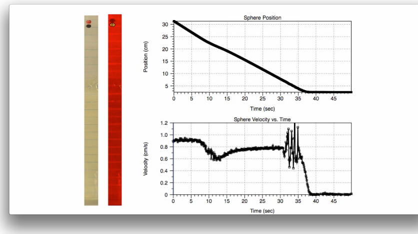


Figure 104: RB1B

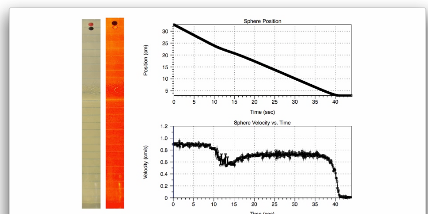


Figure 105: RB2R

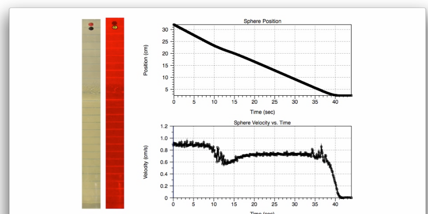


Figure 106: RB2B

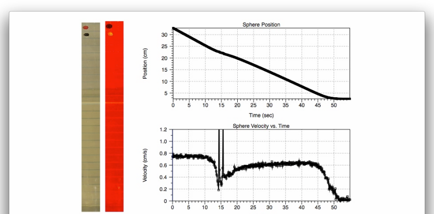


Figure 107: RB3R

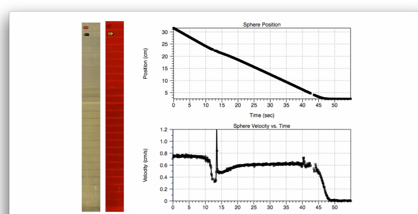


Figure 108: RB3B

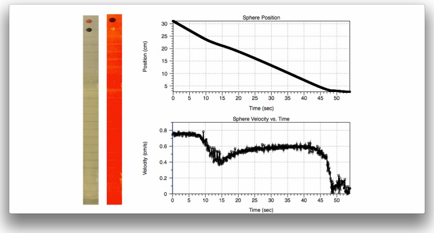


Figure 109: RB4R

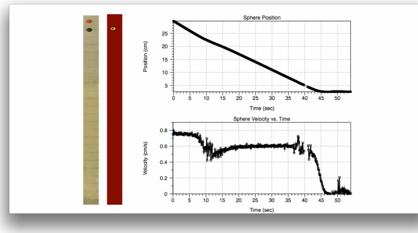


Figure 110: RB4B

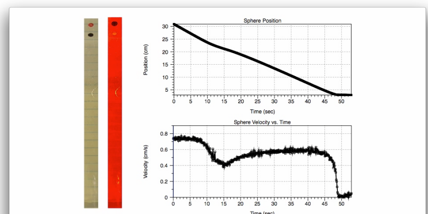


Figure 111: RB5R

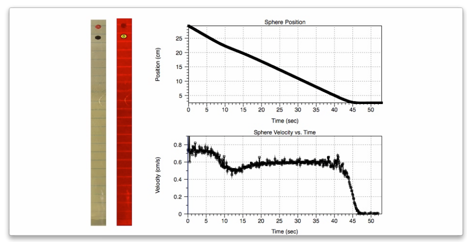


Figure 112: RB5B

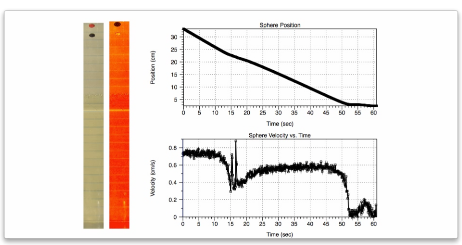


Figure 113: RB6R

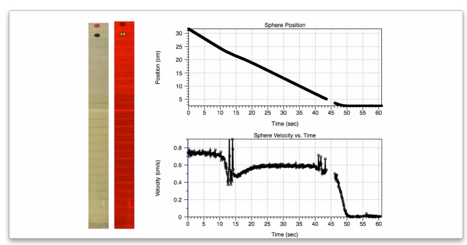


Figure 114: RB6B

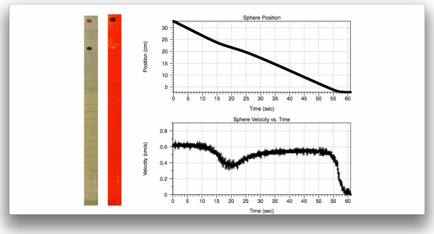


Figure 115: RB7R

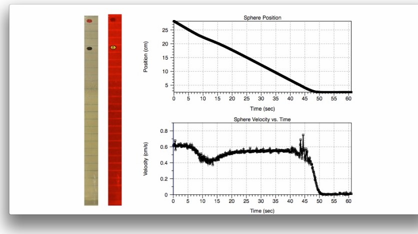


Figure 116: RB7B

**Issues** As mentioned in the note, the temperature should have been taken immediately before the test. Another major issue was not having enough top layer solution. I barely had enough of it to fill a small container with top layer fluid for bead transfer. Also, as is the norm at this point, the method still could use some refinement (quicker, cleaner, less disturbance to the layer). Lastly, issues with aligning the drop, coloration and lighting are always needing improvement. Overall, one of the cleaner test that I have performed.

**Conclusion** The largest conclusion is that the separation effect is still apparent with matched viscosity. Besides that, the experiment reinforced the behavior that we have concluded in experiment 7: shape of the distance plots, velocity in the bottom layer, and the separation effect. The next step will be to have better control for temperature, which means running this experiment in a temperature bath.

# 23 Homogeneous Low Reynolds Fit For Data

I have researched two equations for approximating two spheres falling co-axially, Kynch's and Rushton's.

**Kynch's Approximation:**

Simplifying the Navier-Stokes equation, Kynch presents the following solution for two solid spheres in a low-Reynolds regime:

$$
\left.\begin{aligned}
U_z^A &= V^A + \frac{3b}{4R} V^B \left[ \left(1 + \frac{a^2+b^2}{3R^2}\right) + \sin^2\theta \left(1 - \frac{a^2+b^2}{R^2}\right)\right], \\
U_x^A &= \left(\frac{3b}{4R}\right) V^B \sin\theta \cos\theta \left(1 - \frac{a^2+b^2}{R^2}\right).
\end{aligned}\right\}
$$

Figure 117: **A** is the leading sphere, **B** is the following sphere, **U** is the predicted velocity, **V** is the stokes velocity for a single sphere, **R** is the distance between the two spheres centers, **a** and **b** are the radius' of sphere **A** and **B** respectively, and $\theta$ is the angle from one center to the other (Kynch 198)

We can simplify his equations by assuming: identical spheres, identical stokes velocities, perfectly vertical/stacked orientation (which implies no horizontal movement). Doing so, we get:

$$U = V(1 + \frac{3}{2}(r/R) - (r/R)^3)$$

Where U is the vertical velocity, V is the stokes velocity, r is the radius, and R is the distance between their centers.

**Davie and Rushton's Approximation**

Their simplification to Naiver-Stokes equation produces the following velocity equation for two identical sphere with co-axial orientation in low-Reynolds regimes:

$$\lambda_r = 1 - \frac{3}{4}(r/h) + \frac{9}{16}(r/h)^2 - \frac{27}{64}(r/h)^3 ...$$

(Davie, Ruston 58)

Where $\lambda$ is the ratio of predicted velocity for two spheres, to the stokes velocity of a single sphere (thus non-dimensional), r is the radius, and h is the separation of their centers.

Now that I have ran several experiments in homogeneous fluid (including the top layer in the stratified experiments since they too reach terminal), I can present a dimensionless, regression of all of my cumulative data. Using Mathematica, I computed the following graph:

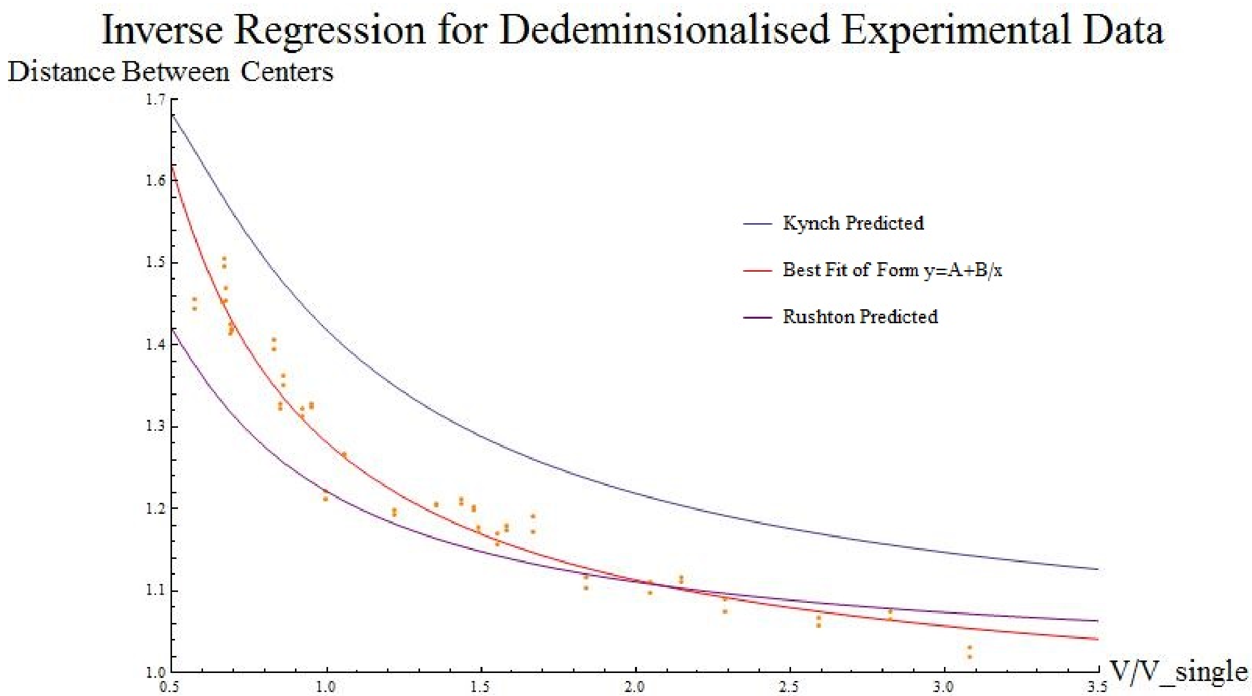



Figure 118: My solution is A=.945 and B=.336 with Standard Dev. $\approx$ 0.009(pred.V/stokesV)

This regression I have produces seems to be reasonable. As distance$\to \infty$, we get 94.5 percent of the actually terminal. Likewise, its standard deviation is multiple orders closer then either of the other equations.

Notice, as we had in our original analysis of the experiments, that the Kynch and Rushton predictions are off. Kynch's equation is acting as an upper bound, and Rushton's is over and under predicting at different spots. For the following sections, I will need a good working theory for predicting these velocities. As of such, I will be using my regression for the purposes of the remainder of this paper. We just need to remember that this is only experimental, and can only apply it to this paper's analysis, and use it for the actually behavior that this paper is after.

# 24 Maximum Distance as a Function of Initial Separation

In experiments 7 and 9, we noticed incredibly regular behavior between the shapes of the distance versus times plots. Although we will need to greatly develop our theory before we can make any mathematically rigorous assertions, we can look at a regression of the maximum distance attained as a function of their initial distance and see if there is some predictive form: linear, exponential, logarithmic, etc.

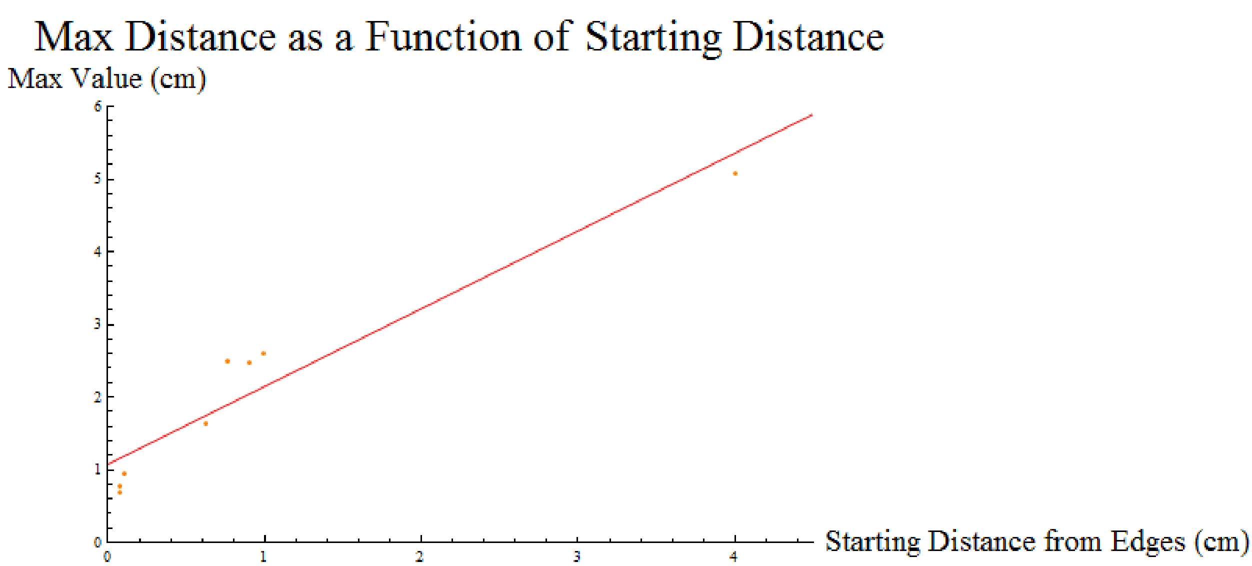



Figure 119: This is a regressional solution of the form y=A+Bx where A=1.078 and B=1.069 with Standard Dev. of $\approx 0.15$cm

Whats particularly amazing about this is that, with a slope of $\approx 1$, the regression suggests that a increase of x centimeters in our initial separation will result in an addition of x centimeters in our maximum separation. It also suggest, with a y-intercept of $\approx 1$, that the spheres will always separate to at least one centimeter apart. Obviously, this data only applies to the very specific parameters of my experiment, and this behavior could be erroneous. But still, it will give us something else to apply our rigorously derived equations to.

# 25 Velocity Fields and Stagnant Points

Now that we have developed a good intuition and have a sizable data set to operate on, it is time to reach the culmination of this thesis with some behavioral mathematical analysis.

By the efforts of (Camassa et al.) we have an analytic model to accurately predict a single sphere in a sharply stratified regime, low-Reynolds. With Kynch's and Rushton's help, we have approximate models to the two sphere, homogeneous low-Reynolds liquid (and for predicting my data set, my regression equation). However, what still escapes us is how the leading sphere and following sphere are interacting in the bottom layer to jointly accelerate and separate (as with the KI), or accelerate and come together (as with the NaCl). Remember, as talked about in the previous section and corroborated by my experiments, two sphere falling co-axially will travel at the same velocity. The knew feature to the experiment that must be accounted for is the sharp stratification and the new forces it introduces into the scenario.

## 25.1 Perturbation Force

One of the most notable forces that arises from stratification is the perturbation force (the force caused by the entrained fluid around the sphere). If not for it, and somehow the spheres smoothly pass through the interface, we would see a velocity plot that transitioned from the top layer terminal, linearly, to the bottom layer terminal and only need a simple Archimedian model. However, as we have seen in our experiments, there is an additional force created by the entrainment of top layer fluid around the outside of the sphere that acts as a life preserver on the much denser bottom layer. This effect has been known to cause the sphere to slow down, or float and even bounce on the interface. This force of entrainment is playing a role in our experiments, and will need to be accounted for.

Again, by the efforts of (Camassa et al.), we have a good understanding of how the perturbation force (the force from the perturbed interface) effects the leading sphere, a situation in which we initially have a flat, undisturbed interface. So lets say this model fully explains our leading sphere. Our following sphere is seeing something completely different, an already contorted/perturbed interface. So, using there force calculation developed by (Camassa et al.) that utilizes greens function, lets observe how the shape of a perturbed interface effects the perturbation force that the following sphere experiences. Here is the equation we will be using:

$$\left.\begin{aligned} \frac{\mathrm{d}Y_3}{\mathrm{d}t}(t;\rho) = V(t;\rho) = (6\pi A\mu K)^{-1}\bigg( m_s g - g\int_{\Omega_s} \rho_0(x_3 + Y_3(t;\rho))\,\mathrm{d}\Omega_s \\ + \int_{\Omega_f} \epsilon G(y,t)\frac{A\rho_{ref}\,g}{4}\left\{\frac{3\left(r^2+y_3^2\right)}{r^3} + \frac{A^2\left(r^2-3y_3^2\right)}{r^5}\right\}\mathrm{d}\Omega_f\bigg), \\ \frac{\partial\rho}{\partial t}(\boldsymbol{x},t) + (\boldsymbol{u}(\boldsymbol{x},t;V) + \boldsymbol{w}(\boldsymbol{x},t;\rho))\cdot\nabla\rho(\boldsymbol{x},t) = 0. \end{aligned}\right\} \tag{3.48}$$

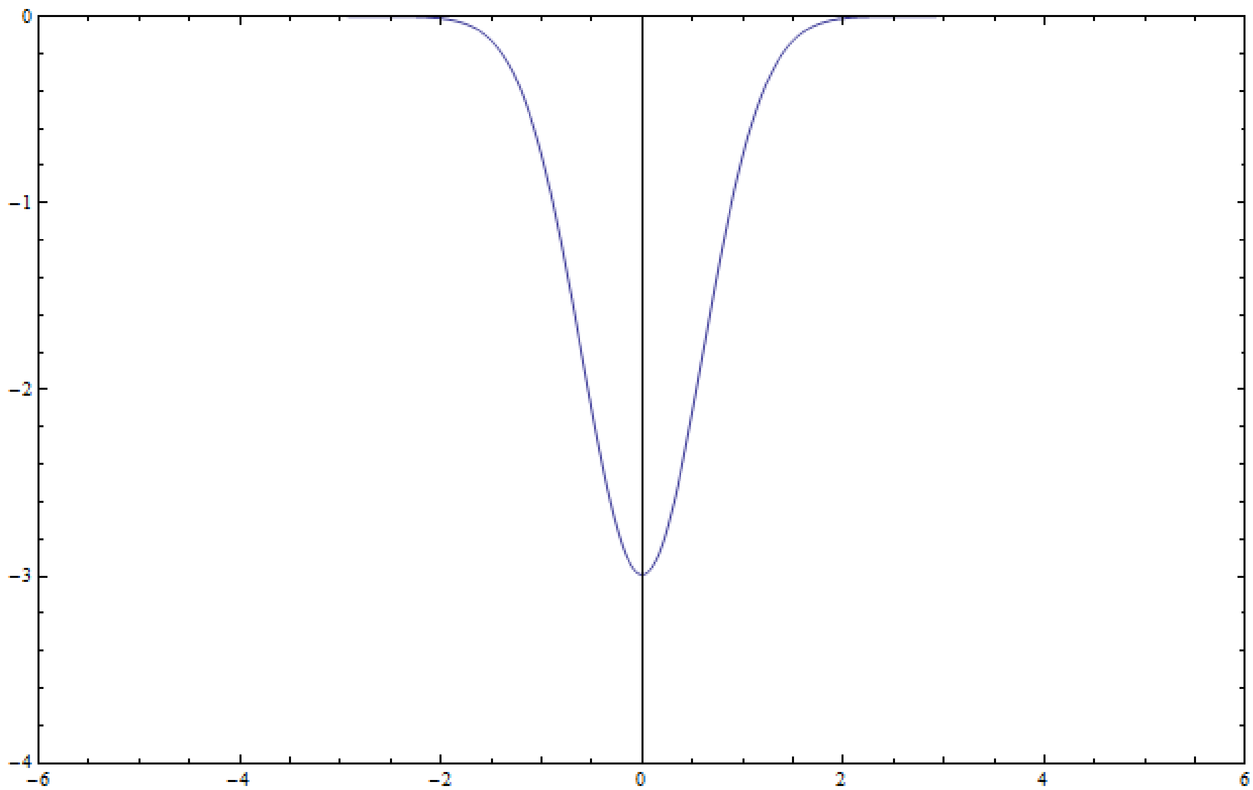
Figure 120: (Camassa et. al, 447)

To simplify the situation, as this is just a behavioral analysis, lets use the Gaussian as our interface (easier to operate on and construct then a discrete one constructed from a video or a simulation). By adjusting the variance parameter, we can approximate how the interface actually deforms.

$$f(x) = ae^{-\frac{bx-\mu}{2\sigma^2}}$$

(Gaussian)
Where $a$ and $b$ are our scaling terms, $\mu$ is the expected value, and $\sigma^2$ is the variance.

Using several videos from experiment 9, we have a good idea of how the interface looks. Specifically, looking at a few frames where the following sphere's center was just passing the interface, I could make out a interface width of approx. 4 centimeters and a height of approx. 3 centimeters (talking about the main bulge with no reflux). This doesn't need to be perfect, we just need to get an approximate shape so I can scale appropriately. To achieve our desired shape. We need to set our parameters to $\mu = 0$, $\sigma^2 = .2$, $a = 1/8$, and $b = 1/3$ to produce a similar interface:




Experimentally, we know that the entrained volume initially increase as the sphere penetrates through the interface, then slowly decreases as the fluid tries to restore itself to the top layer. Our Gaussian needs to reflect that behavior:

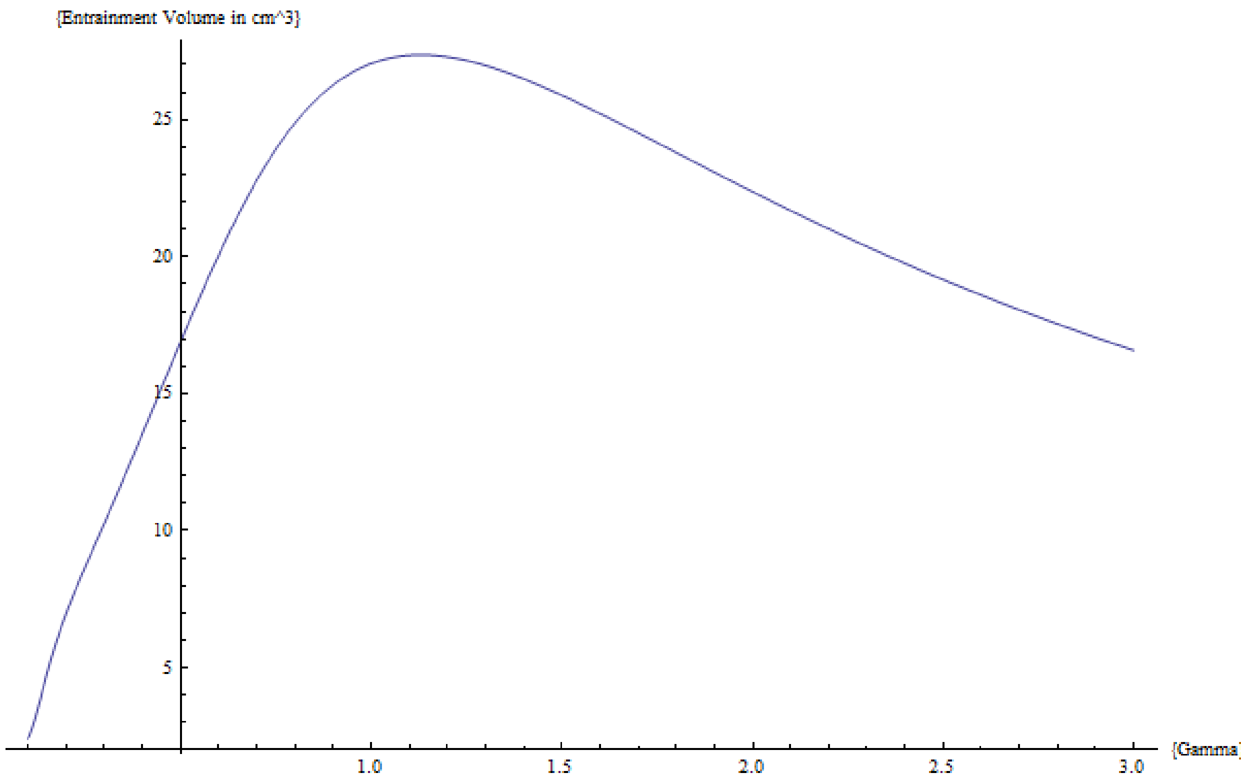
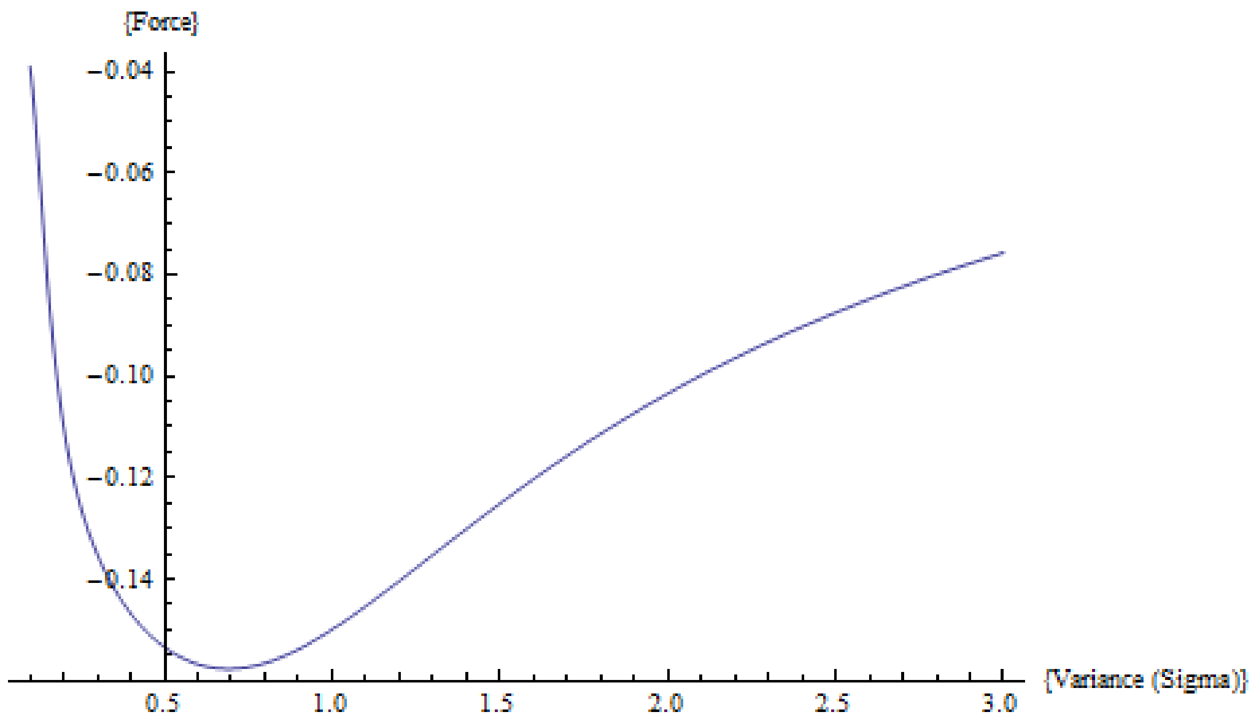



Figure 121: Volume as a function of our variance parameter $\sigma$

Now before we see how this interface interacts with the sphere, we should note the assumptions of this interface: no re-flux (the portion of the interface that extends into the top layer that has a downward restorative force), the Gaussian captures the stretching behavior of the entertainment (we know the volume of entrained fluid slowly diminishes, which we know the Gaussian tries to replicate), the bottom sphere is physically non-existent (we are just concerned with how it stretches the interface and assume that the area it displaces is full of top layer fluid), and that the top sphere's center lies on the interface (we will take the portion of the following sphere that enters the bottom layer into consideration for force). Remember, the point of this is to get a behavioral understanding of the forces at play, and to produce a very rough estimate of the velocity of the following sphere.

Lets begin by seeing what happens to the force it produces as the variances increases (the Gaussian get lower and thinner qualitatively like in experiments:




Now lets use this to predict a velocity with the following parameters (Note that we have inverted the y-axis and positive velocities are down and vice versa):

Density of Top Layer= $1.37419 g/cm^3$
Density of Bottom Layer= $1.42450 g/cm^3$
Density of the Sphere = $2.26 g/cm^3$
Radius of the Sphere (A)= $0.296545 cm$
Tank Radius (R0) = $10.75/2 cm$
Distance to bottom of liquid relative to interface = $-45/2 cm$
Distance to top of liquid relative to interface = $45/2 cm$
gravity = $981 cm/s^2$
Top Layer Viscosity = $23.9264549 Pa.S$
Boundary Condition=$k = (1 - 2.10444(A/R0) + 2.08877(A/R0)^3)^( - 1)$

From this, we the get the velocity of single sphere half way at interface with entertainment force =-0.103717cm/s (its stokes force terminal velocity would have been 0.609764cm/s)

This is obviously wrong, a negative velocity would conclude a bounce which has never occurred in my experiments. However, this is exactly what we would expect in such a qualitative analysis. What we should really focus on is behavior. As our volume increased, we had a increase in the force pushing up on our sphere. Likewise, as our volume began to decrease, we had a decrease in force. Now lets move into finding a more precise calculation of the following sphere velocity by taking into account the flow caused by the leading sphere.

## 25.2 The Faxen Correction

We are beginning to get a fuller picture of our situation. But what is escaping us is how we can have this approach/separation phenomenon. In talking with adviser Camassa and Professor McLaughlin, we concluded that it was completely possible that the approach/separation effect could be a consequence of stagnant points in the flow. Although this has been reported in several papers, (Camassa et. al) produces a good depiction of this flow:

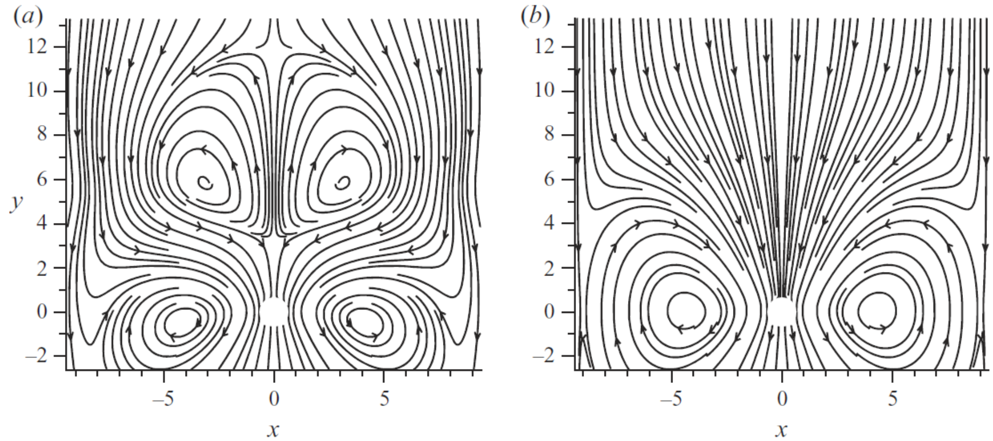



FIGURE 10. In the laboratory frame, the instantaneous streamlines for a sphere of radius 0.635 cm and density 1.5607 g ml$^{-1}$ in a fluid of viscosity $\sim$3.0 Pa s in a cylinder of radius 9.45 cm computed using the full theory with ($a$) fluid of top layer density 1.38147 g ml$^{-1}$ and bottom layer density 1.38506 g ml$^{-1}$ and ($b$) constant density.

Figure 122: Retrieved from (Camassa et. al, 450)

Whats important to note here is the stagnant points (the infinitesimal points where flow is equal to zero). Below the stagnant point, we have a flow that shows particle movement toward the leading sphere, above it, away from the leading sphere. This sounds a lot like our phenomenon!
Now, we need to find a way to relate flow, to the velocity of the following sphere. Faxen's first law fits the bill:

$$F_d = 6\pi\mu\alpha[(1 + \frac{\alpha^2}{6}\nabla^2)u' - (U - u^{\infty})]$$

Where $F_d$ is the force exerted by the fluid on the sphere, $\mu$ is the Newtonian viscosity of the solvent in which the sphere is placed, $\alpha$ is the sphere's radius, $\mathbf{U}$ is the (translational) velocity of the sphere, $\mathbf{u}'$ is the disturbance velocity caused by the other spheres in suspension (not by the background impressed flow), flow evaluated at the sphere centre $\mathbf{u}^{\infty}$ is the background impressed flow (evaluated at the sphere centre and set to zero in our reference).

We also know Archimedian Force from earlier:

$$F_g = (\rho_p - \rho_f)g\frac{4}{3}\pi\alpha^3$$

Where $F_g$ is the Archimedian Force, $\rho_p$ is the density of the sphere, $\rho_f$ is the density of the top layer fluid, $g$ is gravity,$\alpha$ is the radius of the sphere.

By Newton's second law, forces balance if net acceleration is zero. We will assume that over a small enough step size, acceleration$\ll 1$ (Thus this solution is a simplification). Now, since we only have two forces on the sphere (the drag coefficient pushing up and the Archimedian force pulling down), we can equate these two equations and solve for V to get:

$$F_g = F_d$$

$$\Rightarrow U = \frac{2}{9\mu}\alpha^2 g(\rho_p - \rho_f) + u' + \frac{\alpha^2}{6}\nabla^2 u'$$

If we can find: the velocity of the leading sphere (experimentally approximated through regression), an interface shape (from (Camasa et . al) computational simulation for flow), and the flow inside and around the following sphere's center (computationally approximated with (Camassa et.al) model), we can calculate its velocity.

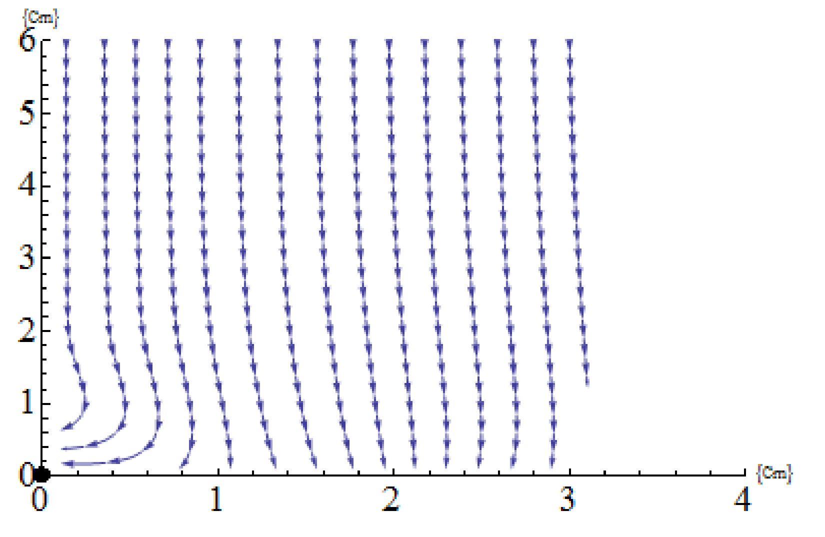
Thanks to (Camassa et. al.), the full flow can be numerically approximated. I will attempt to briefly explain their overall method of derivation. They first split the total flow into two parts: the Stoke's flow for static density, and the density disturbance flow which solves the stoke flow with that extra forcing term from entertainment. They immediately could solve for the static flow since it a well documented. Then they solved the disturbance flow using some careful tricks. They then combine the two flows, by properties of linearity that they conserved, to get the total flow. (Full code and calculation in Mathematica section)

Although we can later use experimental data to get the leading sphere velocity, and Claudia's simulation to get a interface for a full model, let's see if we are on the right track by using our Gaussian interface from the perturbation force calculation and assume certain configurations.

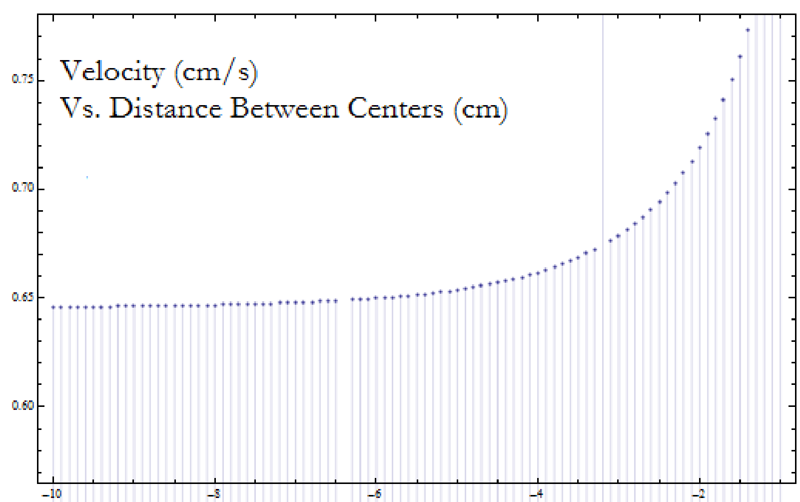
For the following code we will be using these approximate parameters from Experiment 9.
A = 0.259 Radius ($cm$)
mu = 20 Viscosity of top ($Pa.S$)
rhob = 1.37; Density of Bottom ($cm^3$)
rhot = 1.41; Density of Top ($cm^3$)
R0 = 5.4; Radius of tank ($cm$)
iP = TBD; Interface relative to center of sphere ($cm$)
g = 981; Gravity ($cm/s^2$)
V = TBD; Velocity of leading sphere $cm/s$
rhos = 2.29; Density of Spheres $g/cm^3$

**Both Spheres in Upper Layer**
We will put the leading sphere 10 centimeters above the interface which is flat/ nondeformed, and assume its traveling at Stokes Term. velocity for the top layer, 0.6434 cm/s, and look at the predicted velocity of the following sphere.

First, here is what our flow looks like (Note where the black dot is as (0,0) will always be the center of the leading sphere):

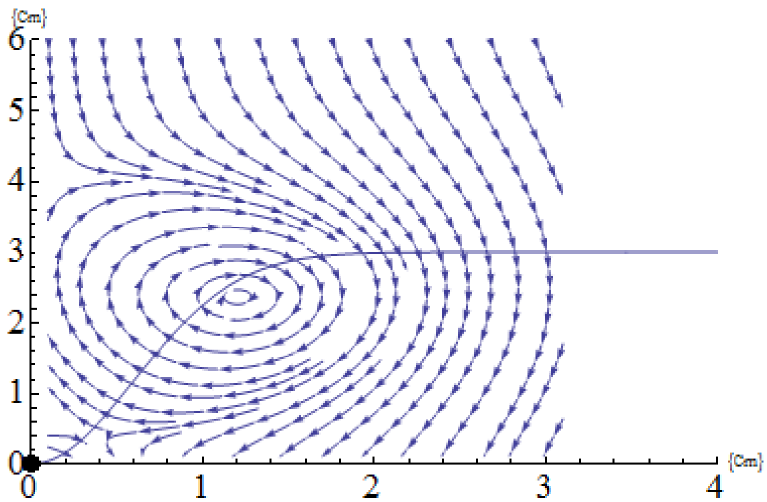



Now here is our velocity for the following sphere as we adjust its position (negative is up).

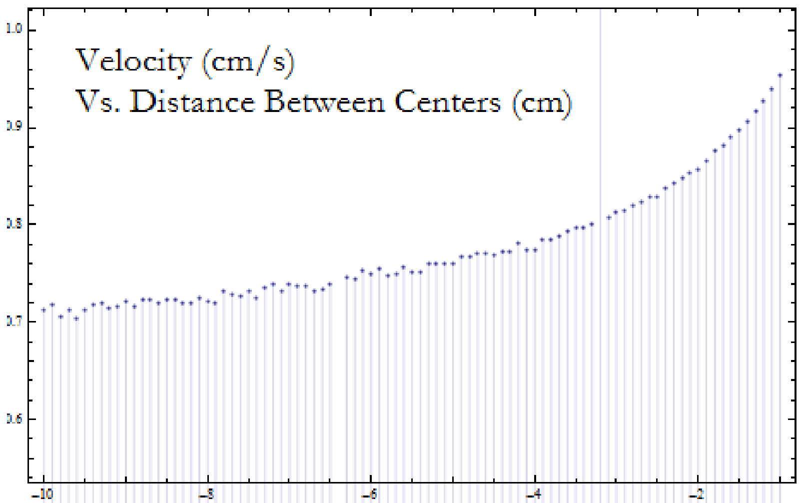



**Leading Sphere just below Interface**
We will put the leading sphere 3 centimeters below the interface, using the interface from the perturbation force, and assume its velocity is .5 (experimentally derived) and look at the predicted velocity of the following sphere.

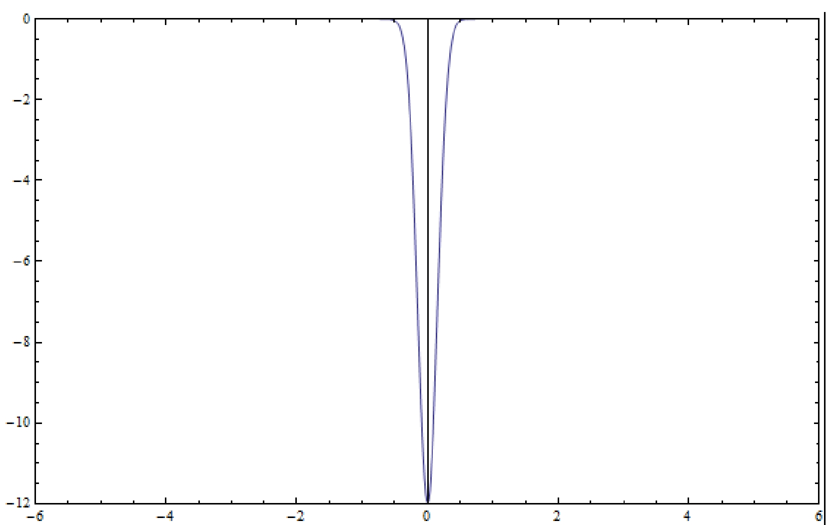
First, here is what our flow looks like:



Now here is our Velocity as we adjust the position of the top sphere.

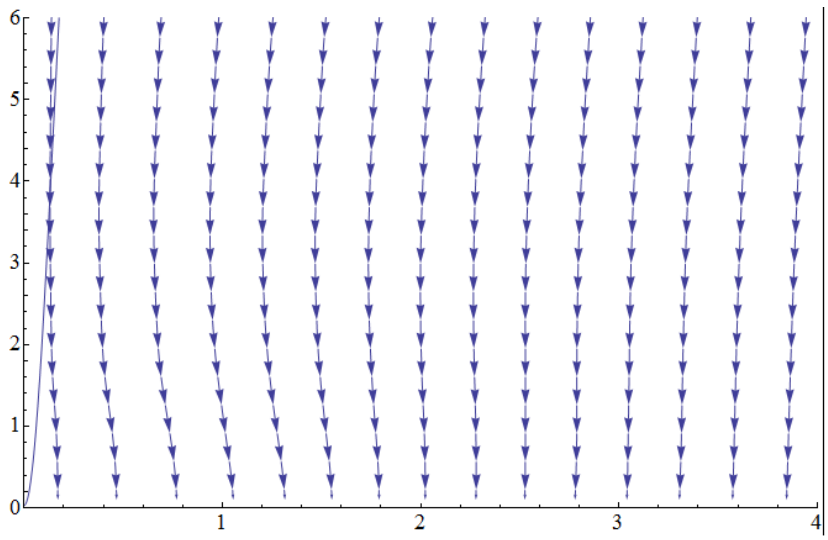



**Leading Sphere far below Interface**
We will put the leading sphere 12 centimeters below the interface using the following interface (constructed by just adjusting $\sigma - .05$):

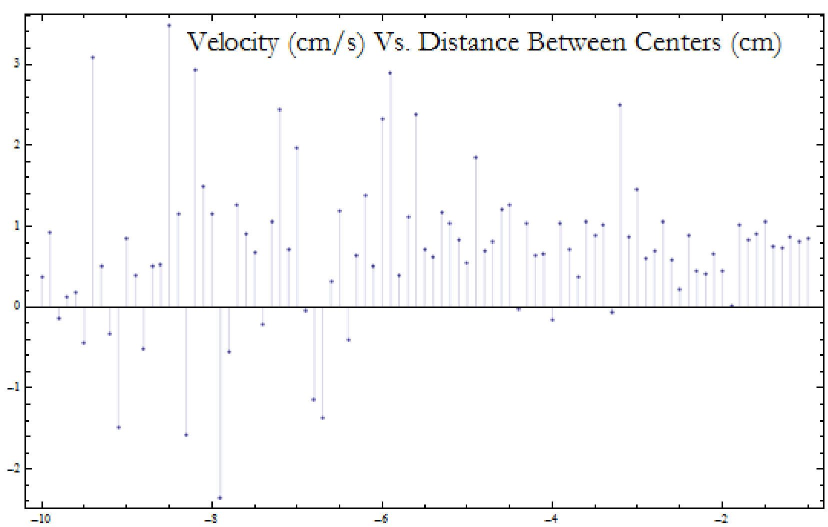


Set the leading sphere speed to the stokes bottom layer terminal velocity speed and look at the predicted velocity of the following sphere.

First, here is what our flow looks like:




Now here is our Velocity as we adjust the position of the top sphere:



The first two velocity versus distance plot results seem incredibly promising, but need to be compared to experiments to offer any final conclusions. For the last plot, we need to do a closer analysis of the data to understand what exactly is happening there. My prediction is that, due to the thinning of the entrainment, we are getting sporadic flows. Possibly taking a more refined Laplacian will fix this? Overall, this presents the most accurate computational modeling of the behavior to this point (the velocities are incredibly reasonable). Now it needs to be compared to my experiments more precisely with a real interface and leading sphere velocity values.

# 26 Conclusion

This paper has been evolving from its inception. Starting with a focus on predicting clouds of particles, we then simplified to two spheres. In a finite Reynolds regime with two spheres, we had issues with "vortex rings", finite Reynolds approximation to the coefficient of drag, and issues with the spheres falling way to fast to track with our cameras. We thus moved to a low-Reynolds regime. There we got stuck trying to test current predictions for two spheres in homogeneous, low-Reynolds regimes because of a broken viscometer. Then finally we moved into a stratified regime with (NaCl) as our salt. Wanting to get a larger effect, we switched to (KI) salt and experienced completely different results. Lastly, we needed to create a similar viscosity between layers to apply (Camassa et al.) equations to our data. Then, with all this data in hand, we were able to go through some calculations and predicative first steps into modeling this system.

Ultimately, these computations are behaviorally predictive, and need to be improved on. Also, there are a wide range of applications that we have yet to explore. So what this paper provides us is the first steps into making a first order model. With this research, we have a far better understanding of an experiment that has never been experimentally explored, a foundation to build and support later research on.

## 26.1 Future work:

**Experimental**
-Using shadow graph, or dying, to track the evolution of an interface.
-Try using different salts.
-Retest with NaCl salt with similar Viscosity
-Retest with a better dropping structure that controls for orientations and separation
-Larger tank for longer videos, less of a boundary effect
-Multiple sharp stratifications in a tank
-Less dense beads to get different times at interface (maybe even a bounce)
-Try other shapes (Cylinder/Ellipoids)
-Explore different sphere sizes/densities relative to each other

**Computational**
-Apply Faxen correction to experimental interface and velocities retrieved from specific experiments.
-See how the following sphere effects the leading sphere using Faxen
-Explore the effects of different densities
-See what happens at extremely small radii
-Explore different sphere sizes/densities relative to each other

**Thank you for taking the time to read this thesis. Look forward to my next publication.**

# 27 Mathematica Code

# Error Propagation

## Predicted Terminal Velocity with Current Measurements

```
Cd[Re_] := 24/Re + (2.6 (Re/5))/(1 + (Re/5)^1.52) + (.411 (Re/263000)^-7.94)/(1 + (Re/263000)^-8) + Re^0.8/461000; (*Mitchell's Coefficient*)
Cd1[Re_] := 24/Re;(*Stoke's Coefficient*)
Cd2[Re_] := 24/Re + 9/2 (*Oseen Coefficient*)
mass = .247033; (*Measured Mass in grams*)
r1 = 2.3433; (*Measured Radius in tenths of an inch*)
r = 2.3433 * .1 * 2.54 / 2; (*Converted Measured Radius in cm*)
Pf = 1.36672; (*Measured Density in g/cm^3*)
u = 21.13; (*Measured Dynamic Viscosity*)


Pp[r_, mass_] := mass/(4/3 Pi * r^3);
RE[r_, u_, V1_] := (V1 * 2 * Pf * r)/u;

2/9 * (Pp[r, mass] - Pf) * 981 r^2/u

FindRoot[8/3 (r * 981)/Cd2[RE[r, u, V1]] (Pp[r, mass] - Pf)/(Pf * V1^2) - 1 == 0, {V1, 1}]

FindRoot[8/3 (r * 981)/Cd[RE[r, u, V1]] (Pp[r, mass] - Pf)/(Pf * V1^2) - 1 == 0, {V1, 1}]
```

```
0.795702
{V1 → 0.791183}
{V1 → 0.795526}
```

## Maximum Terminal Velocity Produced from Error in Instruments

```
Clear[mass, Pf, r, u, Pp];
```

```
mass1 = .247033; (*Measured Mass in grams*)
r2 = 2.3433; (*Measured Radius in tenths of an inch*)
r1 = 2.3433 * .1 * 2.54 / 2; (*Converted Measured Radius in cm*)
Pf1 = 1.36672; (*Measured Density in g/cm^3*)
u1 = 21.13; (*Measured Dynamic Viscosity*)

Error1 = 5 * 10^-4;(*Error for scale as stated by manufacturer*)
Error2 = .001;(*Error from observation using caliper*)
Error3 = .00001;(*Error for density meter as stated by manufacturer*)
Error4 = .005 * u1;(*Error for viscocity as stated by manufacturer*)

mass[error1_] := mass1 + error1;
r[error2_] := (error2 * .1 * 2.54 / 2) + r1;
Pf[error3_] := Pf1 + error3;
u[error4_] := u1 + error4;
Pp[r_, mass_] := mass / (4/3 Pi * r^3);
RE[r_, u_, V1_, Pf_] := (V1 * 2 * Pf * r) / u;

array1 = Table[2/9 * (Pp[r[error2], mass[error1]] - Pf[error3]) * 981 r[error2]^2 / u[error4],
   {error1, -Error1 , Error1, .00001}, {error2, -Error2 , Error2, .0001},
   {error3, Error3 , Error3, .000001}, {error4, 0, Error4, .01}];
stokeMax = Max[array1](*This is the maximum terminal velocity that can
 be produced by Stokes's Coeff. from errors ranging in our uncertainty*)

array2 = Table[8/3 (r[error2] * 981) / Cd2[RE[r[error2], u[error4], V1, Pf[error3]]]
     (Pp[r[error2], mass[error1]] - Pf[error3]) / (V1 * Pf[error3]),
   {error1, -Error1 , Error1, .00001}, {error2, -Error2 , Error2, .0001},
   {error3, -Error3 , Error3, .000001},
   {error4, 0, Error4, .01}, {V1, .5, 2, .01}];
oseenMax = Max[array2](*This is the maximum terminal velocity that can
 be produced by Oseen's Coeff. from errors ranging in our uncertainty*)

array3 = Table[8/3 (r[error2] * 981) / Cd[RE[r[error2], u[error4], V1, Pf[error3]]]
     (Pp[r[error2], mass[error1]] - Pf[error3]) / (V1 * Pf[error3]),
   {error1, -Error1 , Error1, .00001}, {error2, -Error2 , Error2, .0001},
   {error3, -Error3 , Error3, .000001},
```

```
   {error4, 0, Error4, .01}, {V1, .5, 2, .01}];
mitchellMax = Max[array3](*This is the maximum terminal velocity that can
 be produced by Mitchell's Coeff. from errors ranging in our uncertainty*)
```

```
0.801771

0.798907

0.801685
```

# Location of these Errors

```
Position[array1, stokeMax]
Position[array2, oseenMax]
Position[array3, mitchellMax]
```

```
{{101, 1, 1, 1}}

{{101, 1, 1, 1, 1}}

{{101, 1, 1, 1, 1}}
```

# General Equations

```
mass1 = Mass; (*Measured Mass in grams*)
r2 = Radius; (*Measured Radius in tenths of an inch*)
r1 = Radius * .1 * 2.54 / 2; (*Converted Measured Radius in cm*)
Pf1 = DensityFluid; (*Measured Density in g/cm^3*)
u1 = Viscosity; (*Measured Dynamic Viscosity*)

error1 = Er1;(*Error for scale as stated by manufacturer*)
error2 = Er2;(*Error from observation using caliper*)
error3 = Er3;(*Error for density meter as stated by manufacturer*)
error4 = Er4;(*Error for viscocity as stated by manufacturer*)

mass = mass1 + error1;
r = (error2 * .1 * 2.54 / 2) + r1;
Pf = Pf1 + error3;
u = u1 + error4;
```

$$\text{Pp[r\_, mass\_] := } \frac{\text{mass}}{\frac{4}{3}\text{ Pi * r^3}};$$

`u = u1 + error4;`

$$\text{RE[r\_, u\_, V1\_] := } \frac{\text{V1 * 2 * Pf * r}}{\text{u}};$$

$$\frac{2}{9} * (\text{Pp[r, mass]} - \text{Pf}) * 981\ \frac{r^2}{u}$$

$$\text{V[r\_, mass\_, u\_, Pf\_, V1\_] := } \left(\frac{8}{3}\ \frac{(r * 981)}{\text{Cd2[RE[r, u, V1]]}}\ \frac{(\text{Pp[r, mass]} - \text{Pf})}{\text{Pf}}\right)^{1/2};$$

`V[r, mass, u, Pf, V1]`

$$\text{V[r\_, mass\_, u\_, Pf\_, V1\_] := } \left(\frac{8}{3}\ \frac{(r * 981)}{\text{Cd[RE[r, u, V1]]}}\ \frac{(\text{Pp[r, mass]} - \text{Pf})}{\text{Pf}}\right)^{1/2};$$

`V[r, mass, u, Pf, V1]`

$$\left(218\left(-\text{DensityFluid} - \text{Er3} + \frac{3\ (\text{Er1} + \text{Mass})}{4\pi\ (0.127\ \text{Er2} + 0.127\ \text{Radius})^3}\right)(0.127\ \text{Er2} + 0.127\ \text{Radius})^2\right) \Big/ (\text{Er4} + \text{Viscosity})$$

$$2\sqrt{654}\ \sqrt{\left(\left(\left(-\text{DensityFluid} - \text{Er3} + \frac{3\ (\text{Er1} + \text{Mass})}{4\pi\ (0.127\ \text{Er2} + 0.127\ \text{Radius})^3}\right)(0.127\ \text{Er2} + 0.127\ \text{Radius})\right) \Big/ \left((\text{DensityFluid} + \text{Er3})\left(\frac{9}{2} + (12\ (\text{Er4} + \text{Viscosity}))\ / \right.\right.\right.}$$
$$\left.\left.\left.((\text{DensityFluid} + \text{Er3})\ (0.127\ \text{Er2} + 0.127\ \text{Radius})\ \text{V1})\right)\right)\right)$$

$$2\sqrt{654}\,\sqrt{\Bigg(\Bigg(\Bigg(\left(-\text{DensityFluid}-\text{Er3}+\frac{3\,(\text{Er1}+\text{Mass})}{4\,\pi\,(0.127\,\text{Er2}+0.127\,\text{Radius})^{3}}\right)(0.127\,\text{Er2}+0.127\,\text{Radius})\Bigg)\Big/\Big((\text{DensityFluid}+\text{Er3})\,\Big(3.77679\times10^{-6}\,(((\text{DensityFluid}+\text{Er3})\,(0.127\,\text{Er2}+0.127\,\text{Radius})\,\text{V1})\,/\,(\text{Er4}+\text{Viscosity}))^{0.8}+(12\,(\text{Er4}+\text{Viscosity}))\,/\,((\text{DensityFluid}+\text{Er3})\,(0.127\,\text{Er2}+0.127\,\text{Radius})\,\text{V1})+(1.04\,(\text{DensityFluid}+\text{Er3})\,(0.127\,\text{Er2}+0.127\,\text{Radius})\,\text{V1})\Big/\Big((\text{Er4}+\text{Viscosity})\,\Big(1+0.248388\,(((\text{DensityFluid}+\text{Er3})\,(0.127\,\text{Er2}+0.127\,\text{Radius})\,\text{V1})\,/\,(\text{Er4}+\text{Viscosity}))^{1.52}\Big)\Big)+1.81181\times10^{40}\Big/\Big((((\text{DensityFluid}+\text{Er3})\,(0.127\,\text{Er2}+0.127\,\text{Radius})\,\text{V1})\,/\,(\text{Er4}+\text{Viscosity}))^{7.94}\,\Big(1+\big(89\,414\,102\,697\,425\,838\,753\,906\,250\,000\,000\,000\,000\,000\,(\text{Er4}+\text{Viscosity})^{8}\big)\Big/\big((\text{DensityFluid}+\text{Er3})^{8}\,(0.127\,\text{Er2}+0.127\,\text{Radius})^{8}\,\text{V1}^{8}\big)\Big)\Big)\Big)\Big)\Bigg)}$$

# Tools for Calculating Stokes Velocity from Experimental Parameters

$$V_{\mathtt{Terminal}} = \frac{2}{9} * g * r^2 * \frac{(p_{sphere} - p_{fluid})}{mu} * k_{boundary\,constant}$$

$$k = 1 - 2.10444 * (r/r_{tank}) + 2.08877 * (r/r_{tank})^3$$

## Insert Values from Tools Below into the Equation Parameters Below:

```
gravity = 981;
radius = ENTER VALUE;
pSphere = ENTER VALUE;
pFluid = ENTER VALUE;
boundCond = ENTER VALUE;
dViscocity = ENTER VALUE;
termVelocity = 2 / 9 * gravity * (radius)^2 * (pSphere - pFluid) / dViscocity * boundCond
```

### Dynamic Viscocity Calculator:

```
(*Enter Values Here*)
Vis20 = ENTER VALUE;
Vis25 =  ENTER VALUE;
roomTemp = ENTER VALUE;
(*---------------------------------------------------------------------*)
slope = (Vis25 - Vis20) / (25 - 20);
yInt = Vis20 - slope * 20;
V = slope * roomTemp + yInt
(*Viscocity Produced Below*)
```

### Boundary Constant Caculator:

```
(*Enter Values Here*)
rSphere = ENTER VALUE;
rTank = ENTER VALUE;
(*---------------------------------------------------------------------*)
k = 1 - 2.10444 * rSphere / rTank + 2.08877 * (rSphere / rTank)^3
(*Boundary Constant Produced Below*)
```

# Calulcation of Pertubation Force with the Gaussian

```
ClearAll["Global`*"];
(*R - radial component
   Z- vertical component
   2Pi*Integrate[-3A(2*Z^2+R^2)/(4*r^3)+A^3(2Z^2+R^2)/(4*r^5),
    {R,0,R0},{Z,Zmin,Zmax}]- is what we are calculating;*)
r = Sqrt[Z^2 + R^2];(*Equation of our sphere*)

dTop = 1.37419;(*Density of Top*)
dBot = 1.42450;(*Density of Bot*)
rhoS = 2.26;(*Density of Sphere*)
A = 0.296545;(*Radius of our sphere*)
R0 = 10.75 / 2; (*Radius of our tank*)
Zmin = -45 / 2; (*Zmin Distance to bottom of liquid relative to interface*)
Zmax = 45 / 2;(*Zmax Distance to top of liquid relative to interface*)
g = 981;(*Gravity*)
vTop = 23.9264549;(*Viscocity of Top layer*)
k = (1 - 2.10444 (A / R0) + 2.08877 (A / R0) ^3) ^ (-1);(*Boundary Condition*)
μ = 0; (*Where our interface is
 centered at reflexivly in the radial direction*)

(*Our Gaussian*)
Rfun[R_, σ_] := -12 (Sqrt[2 * Pi] * σ) ^-1 * Exp[- (R - μ) ^2 / (2 * σ^2)];

(*Our Half-Sphere that dips into the bottom layer*)
Rfun2[R_] := -Re[Sqrt[A^2 - R^2]];

Clear [Vent, Varch];

(*Velcoity Calculation from JFM*)
Varch = 2 Pi * NIntegrate[- (dTop - dBot) * R *
       (-3 A (2 * Z^2 + R^2) / (4 * r^3) - A^3 (-2 Z^2 + R^2) / (4 * r^5)),
      {R, 0, A}, {Z, Rfun2[R], 0}, AccuracyGoal → 10^(-16)] // Quiet;

Vent[σ_, s_, s2_] := 2 Pi * NIntegrate[- (dTop - dBot) * R *
        (-3 A (2 * Z^2 + R^2) / (4 * r^3) - A^3 (-2 Z^2 + R^2) / (4 * r^5)), {R, 0, R0},
       {Z, Rfun[R * s2, σ] * s, 0}, AccuracyGoal → 10^(-16)] - Varch // Quiet;

(*These Parameters control the Shape of our Gaussian*)
```

```
o = .2;
s = 1 / 8;
s2 = 1 / 3;

(*Stokes Terminal*)
Voft = (6 * Pi * A * vTop * k) ^ (-1) g
   (rhoS * 4 / 3 * Pi * A^3 - ((dTop * 4 / 3 * Pi * A^3) / 2 + (dBot * 4 / 3 * Pi * A^3) / 2));
(*Predicted with Perturbed force*)
Vreal = (6 * Pi * A * vTop * k) ^ (-1) g (rhoS * 4 / 3 * Pi * A^3 -
      ((dTop * 4 / 3 * Pi * A^3) / 2 + (dBot * 4 / 3 * Pi * A^3) / 2) + Vent[o, s, s2]);
```

```
ClearAll["Global`*"];


(*
For the entirety of this code note the following:
   The flow is calculated relative to the center
   of the leading sphere and its center is located at (0,0)
   Also, the axis in flipped horizontally
 (positve velocity is down, negative velocity is up relative to the lab)
*)

A = 0.259;(*Radius*)
mu = 20;(*Visocity of top*)
rhob = 1.37; (*Density of Bottom*)
rhot = 1.41; (*Density of Top*)
R0 = 5.4; (*Radius of tank*)
iP = -3;(*interface relative to center of sphere*)
g = 981;(*Gravity*)
V = .5; (*Velocity of leading sphere*)
rhos = 2.29;(*Density of Spheres*)
```

# Sphere Velocity

```
(*V[Z_]= (1- 2.1044 A/R0 +2.0887 *(A/R0)^3)/(6*Pi *mu*A)*
    ( 4/3 Pi A^3 rhos g - 4/3 Pi A^3 dens0 [Z]) +
   (rhob-rhot)NIntegrate[ R u1z, {theta,0,2*Pi}, {R, 0, R0}, {Z, -Zmin,Zmax}]*)
```

# Stokes Flow u

```
ucorrection = -2.1044 * A / R0 ;
phi = 0;
r = Sqrt[R^2 + Z^2];
u0r = 0;
u1r[R_, Z_] := V (1 - ucorrection)
    ( -3 A R Z / (4 Sqrt[R^2 + Z^2]^3) + 3 A^3 R Z / (4 Sqrt[R^2 + Z^2]^5));

u0z = V;
u1z[R_, Z_] := V ( (1 - ucorrection)
      (-3 A / (4 Sqrt[R^2 + Z^2]) - 3 A Z^2 / (4 Sqrt[R^2 + Z^2]^3) -
        A^3 / (4 Sqrt[R^2 + Z^2]^3) + 3 Z^2 A^3 / (4 Sqrt[R^2 + Z^2]^5))) ;
```

```
(*written in sphere phrame of reference *)

I0 = BesselI[0, lambda R0] ;
I1 = BesselI[1, lambda R0] ;
I2 = BesselI[2, lambda R0] ;

K0 = BesselK[0, lambda R0 ];
K1 = BesselK[1, lambda R0] ;
K2 = BesselK[2, lambda R0] ;


H = A V (3 - (6 + A^2 lambda^2) (K0 I2 + K1 I1)) / (I0 I2 - I1^2);

G = A V (-3 + A^2 lambda^2 (K1 I1 + K2 I0)) / (I0 I2 - I1^2);


J0 = BesselJ[0, I lambda R0];
J1 = BesselJ[1, I lambda R0];

Y0 = BesselK[0, lambda R0];
Y1 = BesselK[1, lambda R0];

urh =
  (lambda R / 2 (H + G) BesselI[0, lambda R] - G * BesselI[1, lambda R]) Sin[lambda Z];

uzh = ( lambda R / 2 ( H + G ) BesselI[1, lambda R] + H BesselI[0, lambda R]) *
   Cos[lambda Z];

(* second reflection *)
u2z[c1_, c2_] :=
  -1 / (2 * Pi) NIntegrate [ uzh /. {R → c1, Z → c2}, {lambda, 0.0000000001, 100}];
u2r[c1_, c2_] := - 1 / (2 * Pi)
   NIntegrate [ urh /. {R → c1, Z → c2}, {lambda, 0.0000000001, 100}];

uz[R_, Z_] := u0z + u1z[R, Z] + u2z[R, Z];
ur[R_, Z_] := u0r + u1r[R, Z] + u2r[R, Z];
```

# Perturbation Flow w

```
Clear[ys1, ys2, ys3, x1, x2, x3, y1, y2, y3]
Rx = Sqrt[x1^2 + x2^2 + x3^2];
Rxy = Sqrt[(x1 - y1)^2 + (x2 - y2)^2 + (x3 - y3)^2];
Ry = Sqrt[y1^2 + y2^2 + y3^2];
Rs = Sqrt[ys1^2 + ys2^2 + ys3^2];
Rxys = Sqrt[(x1 - ys1)^2 + (x2 - ys2)^2 + (x3 - ys3)^2];


Phi = (Ry^2 - A^2) / (2 (Ry^3)) ((3 y3 / (A Rxys)) + (A (x3 - ys3) / (Rxys^3)) +
      (2 y3 / A) (ys1 D[1 / Rxys, x1] + ys2 D[1 / Rxys, x2] + ys3 D[1 / Rxys, x3]) +
      (3 A / Rs) * D[Log[(Rs Rxys + (x1 ys1 + x2 ys2 + x3 ys3 ) - Rs^2) /
          (Rx Rs + (x1 ys1 + x2 ys2 + x3 ys3 ))], ys3]);

ys1 = (A / Ry)^2 y1;
ys2 = (A / Ry)^2 y2;
ys3 = (A / Ry)^2 y3;


W1 = ((x1 - y1) (x3 - y3) / (Rxy^3)) -
    (A / Ry)^3 (x1 - ys1) (x3 - ys3) / (Rxys^3) - (Ry^2 - A^2) / (Ry) *
     (ys1 ys3 / (A^3 Rxys) - A / (Ry^2 Rxys^3) * (ys1 (x3 - ys3) + ys3 (x1 - ys1)) +
       (2 ys1 ys3 (ys1 (x1 - ys1) + ys2 (x2 - ys2) + ys3 (x3 - ys3)) / ((A^3) (Rxys^3)))) -
    (Rx^2 - A^2) * D[Phi, x1];


W2 = ((x2 - y2) (x3 - y3) / (Rxy^3)) -
    (A / Ry)^3 (x2 - ys2) (x3 - ys3) / (Rxys^3) - (Ry^2 - A^2) / (Ry) *
     (((ys2 ys3 / ((A^3) Rxys)) - A / ((Ry^2) ( Rxys^3)) * (ys2 (x3 - ys3) +
            ys3 (x2 - ys2)) + (2 ys2 ys3 (ys1 (x1 - ys1) + ys2 (x2 - ys2) + ys3 (x3 - ys3)) /
            ((A^3) (Rxys^3))))) - (Rx^2 - A^2) * D[Phi, x2];


W3 = (1 / Rxy) - (A / (Ry Rxys)) +
    ((x3 - y3) (x3 - y3) / (Rxy^3)) - (A / Ry)^3 (x3 - ys3) (x3 - ys3) / (Rxys^3) -
    (Ry^2 - A^2) / (Ry) * (((ys3 ys3 / ((A^3) Rxys)) - A / ((Ry^2) ( Rxys^3)) *
          (ys3 (x3 - ys3) + ys3 (x3 - ys3)) + (2 ys3 ys3 (ys1 (x1 - ys1) + ys2 (x2 - ys2) +
               ys3 (x3 - ys3)) / ((A^3) (Rxys^3))))) - (Rx^2 - A^2) * D[Phi, x3];

P = 2 mu * (((x3 - y3) / ( (Rxy^3))) - (A^3 (x3 - ys3) / (Ry^3 Rxys^3)) -
      (Phi + 2 (x1 D[Phi, x1] + x2 D[Phi, x2] + x3 D[Phi, x3])));
```

```
y1 = rho * Cos[theta];
y2 = rho * Sin[theta];
y3 = zeta;

x1 = R * Cos[phi];
x2 = R * Sin[phi];
x3 = Z;

 (*Integrate Over Fluid Domain shell *)

(* write interface curve here *)
(*inter[rho_]: ;*)
σ = .2;
μ = 0;
s = 1 / 8;
s2 = 1 / 3;

Rfun[rho_, σ_] := -12 * s * (Sqrt[2 * Pi] * σ) ^ -1 * Exp[- (rho * s2 - μ) ^2 / (2 * σ^2)];

wr[c1_, c2_] := (rhob - rhot) g / (8 Pi mu) *
    NIntegrate[ rho W1 /. {R → c1, Z → c2} , {theta, 0, 2 * Pi}, {rho, 0, R0},
     {zeta, Rfun[rho, σ] - iP, -iP}, PrecisionGoal → 10^(-16)] // Quiet;

wz[c1_, c2_] := (rhob - rhot) g / (8 Pi mu) *
    NIntegrate[ rho W3 /. {R → c1, Z → c2}, {theta, 0, 2 * Pi}, {rho, 0, R0},
     {zeta, Rfun[rho, σ] - iP, -iP}, PrecisionGoal → 10^(-16)] // Quiet;
```

# Faxen Correction

```
uF[r_, z_] := uz[r, z] + wz[r, z]; (*Full Flow*)
radius = 0.296545; (*radius of sphere*)
dynV = 23.9264549;(*Dynamic Viscosity*)
sPos = -3;(*sphere distance between centers*)
flowC = uF[0, sPos];
rhoS = 2.29; (*Density of Sphere*)
h = .001; (*Step Size*)
laplaceVelocity = (uF[0, sPos + h] - 2 * uF[0, sPos] + uF[0, sPos - h]) / (h) ^2 +
   (uF[0 + h, sPos] - 2 * uF[0, sPos] + uF[0 - h, sPos]) / (h) ^2;
Velocity = (4 / 3 * A^3 * Pi * g * (rhos - rhot) / (6 * Pi * mu * A) - flowC -
    A^2 / 6 * laplaceVelocity) + V (*Velocity with Faxon*)
Velocity = (4 / 3 * A^3 * Pi * g * (rhos - rhot) / (6 * Pi * mu * A)) (*Stokes velocity*)
```

```
0.474456

0.643441
```

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